

# Functional Integration on Constrained Function Spaces

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## Abstract

Analogy with Bayesian inference is used to study constrained physical systems within the context of functional integration. Since functional integrals probe function spaces, both kinematical and dynamical constraints are treated simultaneously and on equal footing. Following the analogy, functional counterparts of conditional and conjugate probability distributions are introduced for integrators and then applied to some well-known examples of constrained functional integrals. The analysis leads to some new functional integration tools and methods. These are utilized to construct a model of the prime counting function as a constrained gamma process.

# 1 Introduction

Kinematical constraints (that is, constraints in the form of boundary conditions) on physical systems modeled by differential equations have been well-studied. Indeed, from the degree of control that has been achieved, it hardly seems necessary to study them further. On the other hand, typical solution methods are based on elementary techniques that rely on simple boundary value matching or symmetry (as in the method of images). But subtleties can arise from more complicated geometries and/or topologies, so it is necessary to extend the elementary methods to include general geometries and topologies — especially in quantum physics.

For example, one extension of elementary methods to general geometries makes use of the generalized Green’s theorem: By formulating the solution of a differential equation in terms of Green’s functions, arbitrary boundary geometries with certain regularity conditions can be treated. As another example, extension to general topologies that may introduce periodicity typically involves the study of functional determinants and spectral theory. In both cases these extensions deal directly with function spaces and the mathematical complexities and subtleties inherent in them.

Similarly, dynamically constrained Hamiltonian systems and their quantization have been — and continue to be — extensively studied for obvious reasons. Solution methods for this constraint type are usually anything but elementary. The vast literature on this topic supports the contention that, here also, the function spaces of the dynamical variables (as opposed to their target manifolds) are of primary importance.

From a function space perspective, the distinction between kinematical constraints and dynamical constraints is unnecessary. Both types can be formulated by posing a restricted or constrained function space: the restriction is often imposed indirectly on a target manifold, and it leads to some kind of set-reduction in some appropriate general function space. Consequently, one can anticipate that function spaces furnish a fruitful arena in which to formulate and study *all* constraints. Moreover, it has long been recognized that functional integration offers reliable if not always acceptably rigorous methods to study function spaces, so it is not surprising that functional integrals have become useful analytical and numerical tools to study complex, constrained physical systems. Importantly, they offer a means to incorporate both kinematical and dynamical constraints under one roof.

There are many references in the physics literature that study constraints in functional integrals; largely utilizing heuristic approaches. The aim of this article is not to supplant those methods — they are useful tools — but to propose a mathematical basis for functional integration on constrained function spaces. The basis is suggested by analogy to Bayesian inference theory, and it affords some guiding principles. With guiding principles in place, useful integration methods can then be developed and tested against known results.

This work will utilize the Cartier/DeWitt-Morette (CDM) scheme as the mathematical foundation of functional integration. A short summary is given in appendix

A, but the reader is encouraged to consult [1]–[3] for background and details. Roughly stated, the CDM scheme uses Fourier duality to define linear integral operators on Banach spaces in terms of bona fide measures on Polish spaces. The as-defined functional integrals, which can be characterized by associated *integrators*, then inherit useful properties through their duality relationship. And these properties can be used to reliably manipulate the functional integrals.<sup>1</sup>

Application of the CDM scheme to unbounded quantum mechanical (QM) systems is well-understood, but how it works under general constraints seems to require new principles. We start by presenting several well-known examples that contain clues to the underlying principles. To begin with, they suggest that constraints add non-dynamical degrees of freedom, and this requires an enlarged function space. Next, the Bayesian analogy suggests the notions of *conditional functional integrals* and *conjugate integrators*. Together with the functional counterpart of the Fubini theorem, these tools enable us to construct and manipulate functional integral representations of constrained QM systems within the CDM scheme.

This is the main idea of the paper: Constrained QM systems require a state space comprising dynamical and non-dynamical degrees of freedom. Singling out a particular system by specifying particular constraints induces a subset of the general state space that we designate as a constrained function space. A specific QM system is then represented by a functional integral based on an appropriate integrator and constrained function space. Finally, the integral over the constrained function space is represented by a functional integral over the full state space but characterized by a *conditional integrator* by defining the functional analogs of marginal and conditional probability densities and using the Bayesian inference analogy to relate them.

Needless to say, the proposed construction must reproduce known QM system results. So some implications and applications of the integral representations are checked against four archetypical classes of constrained QM systems. The four constraint classes can roughly be characterized as i) kinematical, ii) dynamical, iii) periodic, and iv) discontinuous. In particular, we find it necessary to develop the notion of a gamma integrator which is a cousin of the Gaussian integrator. The proposed construction and its associated tools are then applied to particular QM systems. The subsequent efficient derivation of old results illustrates the utility of the new techniques introduced.

As a new application of the formalism we conjecture functional integral representations of some prime counting functions. The functional integrals represent average counting functions, and they give approximations to the exact counting functions that are superior to the conventional approximations. According to the construction, the prime counting functions are modeled as a constrained gamma process —

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<sup>1</sup>Since the CDM scheme is restricted to function spaces whose elements are pointed paths that take their values in some manifold  $\mathbb{X}$ , i.e. maps  $x : [t_a, t_b] \subset \mathbb{R} \rightarrow \mathbb{X}$  with a fixed point  $x(t_a) = x_a \in \mathbb{X}$ , the functional integrals in this article are strictly *path integrals*. However, the CDM scheme can be extended to include more general function spaces (see e.g. [4]) and the guiding principles we identify are not particular to path integrals in this restricted sense. So the term functional integral will be used interchangeably with path integral.

a perspective that may lead to a deeper understanding of the distribution of prime numbers. This is a nice realization of the often observed connection between physics and number theory (see e.g. [5]).

Although the primary focus of the paper is a proposed construction of constrained functional integrals, there are several secondary results obtained along the way that we should point out: 1) The Gaussian and exponential integrators in the CDM scheme are redefined to include a boundary form and a parameter that encodes a *mean* path. The new definitions are more useful in the context of constraints, and they lead naturally to the notion of “effective action”. 2) The complex counterpart of the new Gaussian integrator is defined. Although we do not pursue the idea here, it appears that the complex Gaussian may contain important information regarding the Schrödinger $\leftrightarrow$ diffusion correspondence. Specifically, it might explain when analytic continuation succeeds or fails in this context. 3) In [6] a new integrator within the CDM scheme was introduced based on analogy to a gamma probability distribution. Its utility for incorporating boundary conditions in path integrals was recognized, but its meaning and origin were unclear. Here we learn that the gamma integrator is a natural consequence of the Bayesian analogy. Moreover, the gamma integrator possesses a complex parameter that, when restricted to the natural real numbers, reduces to what can be characterized as a Poisson integrator. In consequence, the ‘propagator’ for a QM system characterized by a gamma integrator yields an equivalent construction of the Poisson functional integral introduced in [1]. 4) Kinematical constraints suggest the notion of a Dirac delta functional on the topological dual of the constraint space. It turns out that these delta functionals are particular types of gamma functional integrals. In fact, gamma functional integrals can be used to define ‘step functionals’ and all their (Gateaux) derivatives. Presumably, one could use such tools to construct a functional analog of distribution theory. 5) Dynamical constraints suggest the notion of Dirac integrators characterized by Dirac delta functionals on the constraint space (as opposed to the dual constraint space). Their heuristic equivalents have long been used to enforce functional constraints — the archetypical example is the Faddeev-Popov method. 6) QM systems with periodic boundary constraints can be efficiently encoded by paths in an appropriate fiber bundle that are constrained to be horizontal. 7) Discontinuous constraints can be handled using the idea of “path decomposition” ([7]-[9]). The functional integral tools lead to a recursive process to calculate propagators on segmented configuration spaces. The recursive process can be evaluated iteratively by the well-known Born approximation, but “path decomposition” suggests a new and potentially useful approximation technique rooted in boundary Green’s functions. More importantly, the functional integral construction show that the propagators associated with such systems are intimately related to Poisson integrators and, hence, gamma integrators.

(A caveat; all variables are assumed unit-less by appropriate normalization for convenience.)

## 2 Motivating examples

### 2.1 Localization

The first class of examples — constrained Feynman path integrals — can be characterized heuristically by the presence of a delta function(al) in the integrand of a path integral. Some particularly prevalent early examples in quantum mechanics of this type were point-to-point transition amplitudes, fixed energy transition amplitudes, and the propagator for a particle on  $S^1$ .

In the CDM scheme, the domain of integration for a path integral is a Banach space of pointed paths  $X_a$ . So point-to-point transition amplitudes are obtained by a suitable delta function in the integrand that ‘pins down’ the loose end of the paths. Standard manipulations ([1]) show that the path integral for quadratic action can be expressed in terms of a restricted domain of paths

$$\begin{aligned} \int_{X_a} \delta(x(t_b), x_b) e^{2\pi i \langle x', x \rangle} \mathcal{D}\omega(x) &=: \int_{X_{a,b}} e^{2\pi i \langle x', x \rangle} \mathcal{D}\omega^{(a,b)}(x) \\ &= \frac{e^{-\pi i W^{(a,b)}(x')}}{\sqrt{\det[i\mathbf{G}(t_b, t_b)]}} \end{aligned} \quad (2.1)$$

where  $\mathbf{G}(t_b, t_b)$  is the covariance associated with the gaussian integrator  $\mathcal{D}\omega(x)$  defined on the space of point paths  $X_a$  and  $W^{(a,b)}$  is the variance associated with the space of point-to-point paths  $X_{a,b}$ .

Aside from the action phase factor and the resulting normalization<sup>2</sup>

$$\int_{X_{a,b}} \mathcal{D}\omega^{(a,b)}(x) = \frac{1}{\sqrt{\det[i\mathbf{G}(t_b, t_b)]}}, \quad (2.2)$$

the new gaussian integrator  $\mathcal{D}\omega^{(a,b)}(x)$  defined on  $X_{a,b}$  is characterized by a *different* covariance  $\mathbf{G}^{(a,b)}(t, s)$  that exhibits the same boundary conditions as paths in  $X_{a,b}$ .

Now consider the other two examples. At the classical level, constraints such as fixed energy and paths on  $S^1$  can be imposed by means of Lagrange multipliers in the classical action. It is then a standard heuristic argument that the Lagrange multiplier constitutes a non-dynamical, path-independent degree of freedom in the path integral that can therefore be integrated out. Essentially, this introduces what can be characterized as a Dirac delta functional. However, to give rigorous meaning to a delta functional, one would need a theory of distributions on  $X_a$ .

An alternative route is to define a Dirac integrator  $\mathcal{D}\delta(x)$  that does the duty of a delta functional. It can be thought of as a limit of a Gaussian integrator with vanishing variance, i.e.  $|W(x')| \rightarrow 0$ . The Dirac integrator reproduces the expected behavior;

$$\int_{X_a} f(x) \mathcal{D}\delta(x) = f(0) \quad (2.3)$$

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<sup>2</sup>This particular normalization is fixed a priori from the choice  $\int_X \mathcal{D}\omega(x) = 1$ .

and

$$\int_{X_a} f(x) \mathcal{D}\delta(M(x)) = \sum_{x_0} \frac{f(x_0)}{\text{Det} M'_{(x_0)}} \quad (2.4)$$

where  $M : X_a \rightarrow X_a$  and  $M(x_0) = 0$ , and it encodes a localization in the integral domain  $X_a$ .

Similarly, an inverse Dirac integrator  $\mathcal{D}\delta^{-1}(x)$  can be formally defined that corresponds to the case  $|W(x')| \rightarrow \infty$  so that

$$\int_{X_a} e^{-2\pi i \langle x', x \rangle} \mathcal{D}\delta^{-1}(x) = \delta(x') . \quad (2.5)$$

This integrator encodes a localization in the dual space  $X'_a$ . In contrast to a Gaussian or Dirac integrator, this type of integrator is not translation invariant;

$$\begin{aligned} \int_{X_a} e^{-2\pi i \langle x', x+x_o \rangle} \mathcal{D}\delta^{-1}(x+x_o) &= \int_{X_a} e^{-2\pi i \langle x'+x'_o, x \rangle} \mathcal{D}\delta^{-1}(x+x_o) = \delta(x') \\ &\Downarrow \\ e^{-2\pi i \langle x'_o, x \rangle} \mathcal{D}\delta^{-1}(x+x_o) &\sim \mathcal{D}\delta^{-1}(x) . \end{aligned} \quad (2.6)$$

Equivalently,

$$\int_{X_a} e^{-2\pi i \langle x', x \rangle} \mathcal{D}\delta^{-1}(x+x_o) = \delta(x' - x'_o) \quad (2.7)$$

where  $\langle x' + x'_o, x \rangle = \langle x', x + x_o \rangle$ .

The salient features of note from these three constrained Feynman path integral examples are: i) constraints are related to a localization in function space (or its dual), ii) constraints are related to a change in covariance and/or mean, and iii) in general the normalization of a constrained integrator is different than the unconstrained integrator.

## 2.2 Quotient spaces

When the target space  $\mathbb{X}$  of the pointed paths  $x : [t_a, t_b] \rightarrow \mathbb{X}$  can be represented as the base space of a principal fiber bundle  $\widetilde{\mathbb{X}}$ , *equivariant* forms on  $\mathbb{X}$  can be expressed in terms of associated forms on  $\widetilde{\mathbb{X}}$ . This technique is essentially symmetry based and allows determination of propagators on multiply connected spaces, orbifolds, compact Lie groups, and homogeneous spaces.

The principal fiber bundle construction is essentially a generalization of the well-known method of images. In practice, the group structure allows the Banach space of paths  $X_a$  to be related to a group decomposition of the Banach space of paths  $\widetilde{X}_a$  where  $\widetilde{x} : [t_a, t_b] \rightarrow \widetilde{\mathbb{X}}$ . In this way, paths taking their values in configuration spaces with non-trivial topology and/or geometry can be treated as paths taking their values in the covering space. This renders a simplified function space — to the

extent allowed by the covering space. In terms of path integrals, the method can be roughly expressed as

$$\int_{X_a} f(x) \mathcal{D}\omega(x) = \int_G \int_{\widetilde{X}_a} \widetilde{f}(\widetilde{x} \cdot g) \mathcal{D}\widetilde{\omega}(\widetilde{x}) \mathcal{D}g . \quad (2.8)$$

where  $G$  is the space of pointed paths  $g : [t_a, t_b] \rightarrow \mathbb{G}$  with  $g(t_a) = g_a$  and  $\mathbb{G}$  the group manifold.

But the functions of interest are equivariant and covariantly constant which means  $\widetilde{x}_a \in \widetilde{\mathbb{X}}$  is parallel transported. Under this ‘constraint’,  $g \mapsto h \in \mathbb{H}_{\widetilde{x}_a}$  the holonomy group at  $\widetilde{x}_a$ , and the integral reduces to the standard result

$$\int_{X_a} f(x) \mathcal{D}\omega(x) = \int_{\mathbb{H}_{\widetilde{x}_a}} \int_{\widetilde{X}_a} \widetilde{f}(\widetilde{x} \cdot h) \mathcal{D}\widetilde{\omega}(\widetilde{x}) dh . \quad (2.9)$$

The point to be made here is that the two function spaces  $\widetilde{X}_a$  and  $X_a$  are related through an integration (and/or summation for multiply connected or discrete holonomy groups). Insofar as finite-dimensional integrals are localized functional integrals, we could loosely say that introducing a constrained integrator on  $\widetilde{X}_a \times G$  along with some localization in  $G$  renders the constrained function space  $X_a$ .

## 2.3 Discontinuous spaces

Our final class of examples is comprised of configuration spaces in which  $x(t)$  experiences some kind of discontinuity. Particular cases include bounded configuration spaces and barrier penetration. The previous two classes of examples gave only a vague hint of how constraints influence a path integral. However, this third class of examples yields valuable clues and insights.

If we believe that a variational principle lies at the heart of the quantum  $\rightarrow$  classical reduction, then we should re-examine the variational problem in the context of constraints. Consider a boundary in configuration space. For point-to-boundary paths, the correct formulation is a variational problem from a fixed initial point to a manifold in the dependent-independent variable space. This type of variational problem introduces a variable end-point that can be interpreted as a non-dynamical dependent variable that encodes the implicit constraints imposed by the configuration space discontinuities and boundaries.

To formulate the variational principle for paths taking their values in a space  $\mathbb{X}$  that intersects a boundary, consider the  $\dim(n+1)$  dependent-independent variable space  $\mathbb{N} = \mathbb{X} \times \mathbb{R}$  with a terminal manifold of dimension  $(n+1) - k$  defined by some set of equations  $\{S_k(x, t) = 0\}$  where  $k \leq n$ ,  $x \in \mathbb{X}$ , and  $t \in [t_a, t_b]$ . Let

$$I(x) = \int_{t_a}^{t_b} F(t, x, \dot{x}) dt$$

be the functional to be analyzed. The extrema of  $I(x)$  solve the variational problem for point-to-boundary paths. In particular, for the case of  $\mathbb{X} = \mathbb{R}^n$ , the variational

problem is solved by the usual Euler equations supplemented by ‘transversality’ conditions (see e.g. [10]).

There are two limiting cases of particular interest. When the terminal manifold in  $\mathbb{N}$  coincides with a boundary (or surface) in  $\mathbb{X}$ , then  $k = 1$  and the transversality conditions reduce to

$$F(t_b, x(t_b), \dot{x}(t_b)) = -\nu \nabla S(x(t_b)) \cdot \dot{\mathbf{x}}(t_b) \quad (2.10)$$

where  $\nu \neq 0$  is a constant and  $x(t_b)$  is on the boundary. For free motion, (2.10) implies that critical paths intersect the boundary transversely.

The other case of interest is when the manifold in the dependent-independent space is ‘horizontal’, i.e.  $x(t_b)$  is fixed and the terminal manifold is a line along the  $t$  direction. This clearly corresponds to a point-to-point transition between two fixed points contained in a bounded region. The terminal manifold is determined by  $k = n$  equations and the transversality conditions yield

$$F(t_b, x(t_b), \dot{x}(t_b)) = \nabla_{\dot{\mathbf{x}}} F(t_b, x(t_b), \dot{x}(t_b)) \cdot \dot{\mathbf{x}}(t_b) \quad (2.11)$$

where  $\dot{\mathbf{e}}$  is a unit vector in the  $\dot{\mathbf{x}}$  direction. If, in particular,  $F = L + E$  where  $L$  is the Lagrangian of an isolated physical system and  $E$  is a constant energy, then this is just the fixed energy constraint  $(\partial L / \partial \dot{x}^i) \dot{x}^i - L = E$ . Consequently, the variational problem in this case is solved by paths with both end-points fixed that have fixed energy ([10]).

There are two lessons to learn from this: i) when boundaries are present, we will need to introduce a non-dynamical degree of freedom, and ii) the boundaries will alter certain expectation values of the paths according to the transversality conditions.

At this point, the nature of the new degree of freedom is obscure. However, if one wants a functional integral to represent the solution of a second order partial differential equation with non-trivial boundary conditions, then a consistent formulation emerges if one is willing to associate the new degree of freedom with a non-Gaussian integrator. It turns out that the new integrator is closely related to a gamma probability distribution in the same way that the Gaussian integrator is related to a Gaussian probability distribution.

The nagging question is “Why a gamma integrator?” The examples have furnished some clues: not surprisingly they point to probability theory. If the answer can be understood, perhaps formulations of path integral representations of more general differential equations will become evident.

### 3 Constraints as conditionals

Consider a physical system with dynamical, topological, and/or geometrical constraints. Postulate that such constraints introduce non-dynamical degrees of freedom. The obvious idea to incorporate these degrees of freedom in a functional integral context is to enlarge the functional space. Consequently, construct  $B \equiv X \times C$  a Banach



product space. The space  $X$  corresponds to what is usually thought of as the space of maps, and  $C$  will be a space of non-dynamical degrees of freedom induced by any constraints. In a probability context, this additional product structure would introduce conditional and marginal distributions. In our context, we expect analogous structures; about which little can be said until the nature of the integrators on  $C$  are understood.

### 3.1 Probability analogy

Here it is fruitful to develop an analogy with Bayesian inference theory.<sup>3</sup> Momentarily pretend that  $B$  is a probability space. Let  $\Theta_X(x)$  and  $\Theta_C(c)$  be the marginal probability distributions on  $X$  and  $C$  respectively. Bayes' theorem implies

$$\begin{aligned}\widetilde{\Theta}_X(x|c) &= \frac{\widetilde{\Theta}_C(c|x)\Theta_X(x)}{\int_X \widetilde{\Theta}_C(c|x)\Theta_X(x) dx} \\ &=: \mathcal{C}(c|x)\Theta_X(x)\end{aligned}\tag{3.1}$$

where  $\widetilde{\Theta}_X$  and  $\widetilde{\Theta}_C$  are conditional probability distributions on  $B$ . This yields insight into the constraint induced normalization noticed in the examples.

This result is not surprising, because a constraint could alternatively be formulated as a map  $M : X \rightarrow Y$  where  $y \in Y$  automatically obeys the constraint. Then change of variable techniques can be used to show the two associated integrators are related through a functional determinant which is essentially  $\mathcal{C}(c|x)$ . This is standard, but it shows that the probability interpretation is consistent (at least with change of variables) and it lends credence to the analogy.

So far, we have only made use of Bayes' theorem. To profit further from the analogy, consider an optical setup where plane monochromatic waves are focused onto an observation screen. We wish to study the nature of the light source by placing various non-conducting apertures between the source and the observation screen. Sooner or later we discover that under mild intensities the irradiance pattern on the observation screen is determined by the mean and covariance of the transmittance at each point in the aperture. Moreover, by changing the wavelength and/or intensity of the source, the resulting irradiance pattern can be predicted.

The Bayesian inferential interpretation of these findings is that the *conditional* probability density or *likelihood*  $\widetilde{\Theta}_X(x|c)$  — which describes the irradiance pattern for a given aperture — can be factored as a product of a functional  $F(x)$  of the transmittance  $x$  and a conditional likelihood that only depends on the mean and covariance of the transmittance. In general, the statement is there exist sufficient statistics  $S_s(x)$  such that

$$\widetilde{\Theta}_X(x|c) = F(x)\widetilde{\Theta}_{S_s(X)}(S_s(x)|c)\tag{3.2}$$

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<sup>3</sup>The classic reference for the probability concepts introduced in this subsection is [11]; especially Ch. 2.

where  $F(x)$  is a functional on  $X$  and  $\widetilde{\Theta_{S_s(X)}(S_s(x)|c)}$  is a likelihood on  $S_s(X) \times C$ . An equivalent statement (by way of Bayes' theorem) is that the conditional probability density  $\widetilde{\Theta_C}(c|x) \propto \Theta_C(c) \widetilde{\Theta_{S_s(X)}(S_s(x)|c)}$  is a functional of sufficient statistics.

There are two key points<sup>4</sup> illuminated by the analogy. The first point is *the effects of a constraint can be inferred from a subset  $S_s(X) \subset X$  given  $\widetilde{\Theta_{S_s(X)}(S_s(x)|c)}$  and the marginal probability density  $\Theta_C(c)$* . And the second is *the marginal and conditional probability distributions on  $C$  belong to the same conjugate family*, i.e.

$$\widetilde{\Theta_C}(c|x) \propto \Theta_C(c) \widetilde{\Theta_{S_s(X)}(S_s(x)|c)} . \quad (3.3)$$

There is great value in these two key points: Given a particular likelihood function and a set of sufficient statistics, the conjugate distributions on  $C$  are highly constrained. In fact, consulting a table of conjugate priors for standard distributions, one can readily find the associated conjugate families.

There are, no doubt, further lessons to be learned about constraints from the probability correspondence, but at this point we leave the analogy and return to the CDM scheme of functional integration (see appendix A for a summary).

### 3.2 Conditional and Conjugate integrators

Return to  $B$  a Banach space of pointed paths, and amend the CDM scheme with the definition<sup>5</sup>

**Definition 3.1** *Let  $B \equiv X \times Y$  be a Banach product space, and let each component Banach space be endowed with CDM scheme data. Define*

$$\Theta_{X|Y}(x|y, x'|y') := \frac{\Theta_B(b, b')}{\Theta_Y(y, y')} = \frac{\Theta_B(b, b')}{\int_X \Theta_B(b, b') \mathcal{D}_{\Theta_X, Z_X} x} \quad (3.4)$$

and

$$Z_{X'|Y'}(x'|y') := \frac{Z_{B'}(b')}{Z_{Y'}(y')} = \frac{Z_{B'}(b')}{\int_{X'} Z_{B'}(b') d\mu_{X'}(x')} . \quad (3.5)$$

*These two functionals define an associated conditional integrator by*

$$\int_B \Theta_{X|Y}(x|y, x'|y') \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y := Z_{X'|Y'}(x'|y') . \quad (3.6)$$

*The space  $\mathcal{F}_{X|Y}(B)$  of conditional integrable functionals consists of functionals defined by*

$$F_\mu(x|y) := \int_{B'} \Theta_{X|Y}(x|y, x'|y') d\mu(x'|y') = \int_{B'} \frac{\Theta_B(b, b')}{\Theta_Y(y, y')} d\mu(x'|y') \quad (3.7)$$

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<sup>4</sup>These points assume the system is not driven ‘too hard’ so that the probability distribution that characterizes the system does not change during trials or observations.

<sup>5</sup>The use of  $\Theta$  in both the probability and functional integral context is meant to be suggestive, but it should be kept in mind that the same symbol is referring to two distinct objects that should not be confused.

where  $\mu(x'|y')$  is a conditional measure<sup>6</sup> on  $B'$ . And the linear integral operator on  $\mathcal{F}_{X|Y}(B)$  is given by

$$\int_B F_\mu(x|y) \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y = \int_{B'} Z_{X'|Y'}(x'|y') d\mu(x'|y') . \quad (3.8)$$

**Proposition 3.1**

$$\begin{aligned} \int_B \Theta_{X|Y}(x|y, x'|y') \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y &= \frac{1}{Z_{Y'}(y')} \int_B \Theta_B(b, b') \mathcal{D}_{\Theta_B, Z_B} b \\ &\Downarrow \\ \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y &\sim \frac{\Theta_Y(y, \cdot)}{Z_{Y'}(\cdot)} \mathcal{D}_{\Theta_B, Z_B} b \end{aligned} \quad (3.9)$$

In particular, since the integrator relation holds for any  $y' \in Y'$ ,

$$\int_B F_\mu(x|y) \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y = \frac{1}{Z_{Y'}(y')} \int_B F_\mu(x|y) \Theta_Y(y, y') \mathcal{D}_{\Theta_B, Z_B} b \quad (3.10)$$

most often with  $\langle y', y \rangle = S_s(y)$  or  $\langle y', y \rangle = 0$  for all  $y \in Y$ .

*Proof.* The proof follows immediately from definition 3.1 and the relevant CDM definitions.  $\square$

Evidently expressing integrals like  $\int_B F(b) \mathcal{D}b$  when  $B$  is a product space requires knowledge of the ‘marginal’ and ‘conditional’ integrators on the component spaces. Of course, when elements in  $X$  and  $Y$  are independent, the conditional integrator on  $B$  reduces to a simple product of standard integrators on  $X$  and  $Y$ . But we anticipate that constraints induce a dependence between elements in  $X$  and  $Y$ .

Now specialize to the case when  $Y$  represents non-dynamical degrees of freedom — perhaps due to constraints. As suggested by the optical diffraction example, postulate that the physical system is described by ‘sufficient statistics’<sup>7</sup> and that  $\Theta_Y$  and  $\Theta_{Y|X}$  belong to the same conjugate family. Then knowledge of the ‘likelihood’ functional  $\Theta_{X|Y}(x|y, x'|y')$  implies knowledge of the conjugate family of  $\Theta_Y(y, y')$ . Consequently, the heuristic integral  $\int_B F(b) \mathcal{D}b$  will be well defined in terms of known functionals.

Accordingly, the Bayesian analogy suggests the definition:

**Definition 3.2** *Conjugate integrators are characterized by*

$$\Theta_{Y|X}(y|x, \cdot) \propto \Theta_{S_s(X)|Y}(S_s(x)|y, \cdot) \Theta_Y(y, \cdot) \quad (3.11)$$

where

$$\int_Y \Theta_Y(y, y') \mathcal{D}_{\Theta_Y, Z_Y} y = Z_Y(y') \quad (3.12)$$

and the proportionality is fixed by normalization.

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<sup>6</sup>The conditional measure is well defined as the restriction of  $\mu$  to the appropriate sub- $\sigma$ -algebra over  $B'$ .

<sup>7</sup>In a functional integral context, ‘sufficient statistics’ is interpreted naturally as a localization in the space of paths precipitated by some constraint.

Note that this implies (by Bayes' theorem)

$$\Theta_{X|Y}(x|y, \cdot) \propto \Theta_B((S_s(y), x), \cdot) . \quad (3.13)$$

This property suggests the solution strategy

$$\begin{aligned} \int_{\tilde{X}} F_\mu(\tilde{x}) \mathcal{D}_{\Theta_{\tilde{X}}, Z_{\tilde{X}}} \tilde{x} &:= \int_B F_\mu(x|y) \mathcal{D}_{\Theta_{X|Y}, Z_{X|Y}} x|y \\ &=: \int_B \tilde{F}_\mu(y, x, \cdot) \Theta_Y(y, \cdot) \mathcal{D}_{\Theta_B, Z_B} b \\ &= \int_X \left[ \int_Y \tilde{F}_\mu(y, x, \cdot) \Theta_Y(y, \cdot) \mathcal{D}_{\Theta_Y, Z_Y} y \right] \mathcal{D}_{\Theta_X, Z_X} x \\ &= \int_X \tilde{G}_\mu(S_s(y), x) \mathcal{D}_{\Theta_X, Z_X} x \end{aligned} \quad (3.14)$$

where the integral on the left is interpreted as a constrained functional integral, i.e. an integral over the constrained function space  $\tilde{X}$ , the third line employs functional Fubini (Prop. A.3 in [6]), and  $\tilde{G}_\mu$  can be interpreted as a constrained functional depending on the constraints only through sufficient statistics. This is the functional integral analog of (3.1). Notice that the statement is equally valid with  $X \leftrightarrow Y$  if one knows some  $S_s(X)$ ; hence suggesting an alternative solution strategy

$$\begin{aligned} \int_{\tilde{X}} F_\mu(\tilde{x}) \mathcal{D}_{\Theta_{\tilde{X}}, Z_{\tilde{X}}} \tilde{x} &:= \int_B F_\mu(y|x) \mathcal{D}_{\Theta_{Y|X}, Z_{Y|X}} y|x \\ &=: \int_B \tilde{F}_\mu(x, y, \cdot) \Theta_X(x, \cdot) \mathcal{D}_{\Theta_B, Z_B} b \\ &= \int_Y \left[ \int_X \tilde{F}_\mu(x, y, \cdot) \Theta_X(x, \cdot) \mathcal{D}_{\Theta_X, Z_X} x \right] \mathcal{D}_{\Theta_Y, Z_Y} y \\ &= \int_Y \tilde{H}_\mu(S_s(x), y) \mathcal{D}_{\Theta_Y, Z_Y} y . \end{aligned} \quad (3.15)$$

Both strategies will be employed in the next section.

The important point worth emphasizing is that  $\Theta_{Y|X}$  and  $\Theta_Y$  belong to the same family of integrators, and they are simply related through the sufficient statistics that describe the integrator on  $X$ . This quickly narrows the search for an integrator associated with a particular constraint.

However natural the probability analogy may seem, the usefulness of the defined functional integrals rests on their efficacy — which in turn depends on establishing physically relevant conditional integrators. We describe in detail four particularly prevalent integrator families in Appendix B and use them in the next section to re-examine the motivating examples of §2.

## 4 Examples revisited

The value of the previous section is perhaps best appreciated by application to the familiar examples from QM outline in §2.

Eigenfunctions of position observables in quantum mechanics (for the semiclassical approximation) are characterized by their mean and covariance. When viewed as functions of time, they become paths in some configuration space. The semiclassical quantum mechanics can be described in terms of functional integrals with Gaussian integrators on the space of paths characterized by an assumed mean and covariance.

Applying the previous section to the case  $Y_a \equiv C_a$ , it is clear that the imposition of constraints in QM can change the quadratic form and/or the mean associated with an unconstrained Gaussian integrator. The notion of conjugate integrators enables one to construct the ‘marginal’ and ‘conditional’ integrators associated with those constraints, and hence make sense of constrained functional integrals. We will see that the type of conjugate family depends on the type of constraint.

## 4.1 Localization

### 4.1.1 Delta functional

**Definition 4.1** *Let the space of  $L^{2,1}$  pointed paths  $x : [t_a, t_b] \rightarrow \mathbb{X}$  be endowed with a Gaussian integrator  $\mathcal{D}\omega_{\bar{x},Q}(x)$ . A delta functional on  $C'_a$  is a constraint that alters the covariance but not the mean of  $x \in X_a$  relative to  $\mathcal{D}\omega_{\bar{x},Q}(x)$ . (Here  $C'_a$  is the topological dual Banach space of the space of pointed paths in the constraint function space  $C_a$ .)*

The conjugate family associated with a normal distribution of known mean and unknown covariance is a gamma distribution, so the integrator associated with a delta functional is expected to be a gamma integrator.<sup>8</sup>

As motivation, consider the lower gamma integrator family. Put  $\alpha = 1$ , and let  $L : T_a \rightarrow \mathbb{R}^n$  with  $\langle \beta', L\tau \rangle \mapsto 2\pi i \boldsymbol{\lambda} \cdot \mathbf{u}$  and  $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$ . Then formally,

$$\int_{T_a} \mathcal{D}\gamma_{1,\beta',\infty}(\tau) \longrightarrow \int_{\mathbb{R}^n} e^{-2\pi i \boldsymbol{\lambda} \cdot \mathbf{u}} d\mathbf{u} = \delta(\boldsymbol{\lambda}) . \quad (4.1)$$

with  $\boldsymbol{\lambda} \in \mathbb{R}^n$  and the integral over  $\mathbb{R}^n$  understood as an inverse Fourier transform. On the other hand,

$$\int_{T_a} \mathcal{D}\gamma_{1,\beta',\tau_{\partial}}(\tau) = \frac{\gamma(1, \overline{\tau_{\partial}})}{\text{Det}(\beta')} = \frac{1 - e^{-\overline{\tau_{\partial}}}}{\text{Det}(\beta')} , \quad (4.2)$$

and the integrator  $\mathcal{D}\gamma_{\alpha,\beta',\infty}(\tau)$  should be understood as a limit;

$$\int_{T_a} \mathcal{D}\gamma_{\alpha,\beta',\infty}(\tau) := \lim_{|\overline{\tau_{\partial}}| \rightarrow \infty} \int \mathcal{D}\gamma_{\alpha,\beta',\tau_{\partial}}(\tau) . \quad (4.3)$$

So when  $\overline{\tau_{\partial}}$  is imaginary,  $\mathcal{D}\gamma_{1,i\beta',\infty}(\tau)$  can be thought of as the functional analog of a two-sided Laplace transform implying

$$\int_{T_a} \mathcal{D}\gamma_{1,i\beta',\infty}(\tau) = \lim_{|\overline{\tau_{\partial}}| \rightarrow \infty} \frac{e^{i\overline{\tau_{\partial}}} - e^{-i\overline{\tau_{\partial}}}}{\text{Det}(i\beta')} . \quad (4.4)$$

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<sup>8</sup>The gamma family is also conjugate for exponential-type distributions so the following analysis holds for action functionals that are not quadratic.

This justifies the definition:

**Definition 4.2** Let  $\text{Det}(ic') = 0$ . A delta functional<sup>9</sup> on  $C'_a$  is defined by

$$\delta(c') := \frac{1}{\Gamma(1)} \int_{C_a} \mathcal{D}\gamma_{1,ic',\infty}(c) \quad (4.5)$$

and a Heaviside step functional by

$$\theta(c') := \frac{1}{\Gamma(0)} \int_{C_a} \mathcal{D}\gamma_{0,ic',\infty}(c) . \quad (4.6)$$

The delta functional vanishes unless  $\langle c', c \rangle = 0$  for some  $c \neq 0$ , i.e. it picks out the zero modes of  $c'$ . The step functional vanishes unless  $|\langle c', c \rangle| < c_o$  for some fiducial  $c_o \in \mathbb{C}$ .

Remark that this definition suggests the characterization

$$\delta^{(\alpha-1)}(c') = \frac{1}{\Gamma(\alpha)} \int_{C_a} \mathcal{D}\gamma_{\alpha,ic',\infty}(c) . \quad (4.7)$$

Evidently gamma integrators and their associated functional integrals could be used as a basis for a theory of what might be called ‘distributionals’.

Return to the (free) QM point-to-point propagator in  $\mathbb{R}^n$ . Impose the constraint  $\delta(x(t_b) - x_b)$  where  $x_b := \bar{x}(t_b)$  with the choice

$$\begin{aligned} -\langle \beta(x)', c \rangle &:= 2\pi i \int_{t_a}^{t_b} [x(t_b) - x_b] \cdot c(t) dt \\ &= 2\pi i [x(t_b) - x_b] \cdot \int_{t_a}^{t_b} c(t) dt \\ &=: 2\pi i [x(t_b) - x_b] \cdot \bar{c} \end{aligned} \quad (4.8)$$

where  $c(t) \in \mathbb{R}^n$  and  $\beta : X_a \rightarrow C'_a$ . Clearly, the integral in definition 4.2 reduces to a simple delta function on  $\mathbb{R}^n$  in this case, and we can write

$$\int_{X_a} \mathcal{D}\omega_{\bar{x},Q(a,b)}(x) := \int_{X_a \times C_a} e^{2\pi i [x(t_b) - x_b] \cdot \int_{t_a}^{t_b} c(t) dt} \mathcal{D}_{1,0,\infty}(c) \mathcal{D}\omega_{\bar{x},Q}(x) \quad (4.9)$$

where the right-hand side is to be interpreted as the integral of a conditional integrator on  $X_a \times C_a$ .

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<sup>9</sup>Since  $C'_a$  is a polish space, the delta functional can be interpreted in a measure theoretic sense as the complex Borel measure associated with the identity element in the Banach space  $\mathcal{F}(C_a)$ , i.e.  $\text{Id} = F_\mu(c) = \int_{C'_a} e^{i\langle c', c \rangle} d\mu(c') \quad \forall c \in C_a$  (see appendix).

Doing the calculation<sup>10</sup> (for  $s = i$ );

$$\begin{aligned}
\int_{X_a} \mathcal{D}\omega_{\bar{x}, Q(a,b)}(x) &= \int_{X_a \times C_a} c e^{-\langle \beta(x)', c \rangle} \mathcal{D}c \mathcal{D}\omega_{\bar{x}, Q}(x) \\
&= \int_{C_a} \int_{X_a} c e^{2\pi i \langle \langle \delta_{t_b}, (x-\bar{x}) \rangle', c \rangle} \mathcal{D}\omega_{\bar{x}, Q}(x) \mathcal{D}c \\
&= \int_{C_a} c \int_{X_0} e^{2\pi i \langle \langle \delta_{t_b}, \bar{x} \rangle', c \rangle} \mathcal{D}\omega_{0, Q}(\bar{x}) \mathcal{D}c \\
&= e^{\pi i B(\bar{x}(t_b))} \sqrt{\det [i \mathbf{G}(t_b, t_b)]} \int_{C_a} c e^{\langle \pi i W(\delta_{t_b})', c \rangle} \mathcal{D}c \\
&= e^{\pi i B(\bar{x}(t_b))} \sqrt{\det [i \mathbf{G}(t_b, t_b)]} \frac{1}{(\det [i \mathbf{G}(t_b, t_b)])} \\
&= \frac{e^{\pi i B(\bar{x}(t_b))}}{\sqrt{\det [i \mathbf{G}(t_b, t_b)]}} \tag{4.10}
\end{aligned}$$

where we used functional Fubini, the boundary form in the fourth line appears because  $Q$  on the Hilbert space  $X_0$  with the new boundary conditions is not self-adjoint, the fifth line follows since  $\text{Det}(iW(\delta_{t_b})) \neq 0$ , and  $\bar{x}$  in the last line is the mean (actually critical in this case) path with boundary conditions  $\bar{x}(t_a) = x_a$  and  $\bar{x}(t_b) = x_b$ . That is,

$$\bar{x}(t) = \frac{x_a(t_b - t) + x_b(t - t_a)}{(t_b - t_a)}. \tag{4.11}$$

It should be emphasized that  $\mathbf{G}(t_b, t_b)$  is the covariance matrix associated with the original boundary conditions  $\bar{x}(t_a) = x_a$  and  $\dot{\bar{x}}(t_b) = \dot{x}_b$ .<sup>11</sup> Also note that, since the mean path is non-vanishing, there is a phase associated with the boundary functional  $B$  (defined in Appendix B) that is not evident in the unbounded case represented by (2.2).

But this is still not the propagator. There may be more than one critical path, and in general there may be more than one mean path relative to an action functional

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<sup>10</sup>Although this calculation has been done many times by time slicing, semi-classical, or linear mapping techniques; the point of repeating it here is to demonstrate that the as-defined functional integral tools allow the calculation to be carried out entirely at the level of the function space — the target manifold only makes an appearance through the boundary conditions imposed on the mean and covariance and the regularization/normalization of the functional determinant.

<sup>11</sup>Significantly, we did not have to expand about the critical path to do the calculation. Including the mean and boundary form in the definition of the integrator automatically handles this for us. But it does more. It tells us that, when  $Q \rightarrow S$ , the ‘sufficient statistic’ of import is not the critical path but the mean path. So, for example, the semi-classical expansion in terms of the mean path automatically accounts for self-interactions. In other words, once the mean is known, the Feynman diagram procedure (now without loop diagrams) is a way to estimate the moments of the integrator  $\mathcal{D}\omega_{\bar{x}, S}$ . As already remarked in Appendix B, this is the essence of the effective action approach in quantum field theory.

$S$  with the appropriate boundary conditions. Hence, for self-adjoint  $D$  on  $X_a$ ,

$$K_Q(x_a, x_b) = \sum_{\bar{x}} \frac{e^{\pi i B(\bar{x}(t_b))}}{\sqrt{\det [i \mathbf{G}(t_b, t_b)]}}. \quad (4.12)$$

The subscript  $Q$  has been included here to emphasize that the propagator is determined by a covariance associated with specific boundary conditions, and it is a sum over  $\bar{x}$  of *all* Gaussian integrators with  $\bar{x}$  having the appropriate boundary conditions. This will be a recurring theme: propagators are represented by a sum over relevant parameters of an integrator family.

The point-to-point propagator is a specialization of the more general integral (assuming convergence of the integral)

$$K_S(x_a, x_b) := \sum_{\bar{x}} \int_{X_a} \delta(\beta(x)') \mathcal{D}\omega_{\bar{x}, S}(x) = \int_{X_a} \delta(\beta(x)') \sum_{\bar{x}} \mathcal{D}\omega_{\bar{x}, S}(x) \quad (4.13)$$

where  $\text{Det } \beta(x)' = 0$  and the mean paths satisfy associated boundary conditions. It is to be understood as an integral over a conditional integrator on  $X_a \times C_a$ .

According to the functional Fubini, the order of integration can be interchanged for product integrators (the analog of independent joint distributions). Then, since the gamma family is conjugate for Gaussian likelihood functionals, the integral over  $X_a$  will yield another gamma integrator — in which case the integral with respect to  $\mathcal{D}c$  is well defined. Explicitly,

$$\begin{aligned} \int_{X_a} \delta(\beta(x)') \mathcal{D}\omega_{\bar{x}, Q}(x) &= \int_{B_a} \mathcal{D}\omega_{\bar{x}, Q}(x) \mathcal{D}\gamma_{1, i\beta(x)', \infty}(c) \\ &=: \int_{C_a} \langle e^{-i\langle \beta(x)', c \rangle} \rangle_{\omega_{\bar{x}, Q}} \mathcal{D}\gamma_{1, 0, \infty}(c) \end{aligned} \quad (4.14)$$

and the Gaussian expectation of  $\exp\{-i\langle \beta(x)', c \rangle\}$  must lie in the family of gamma integrators. Likewise,  $\langle \exp\{-\pi/s Q(x - \bar{x})\} \rangle_{\gamma_{1, i\beta(x)', \infty}}$  must lie in the family of Gaussian integrators — which brings us to the next subsection.

#### 4.1.2 Uniform functional

**Definition 4.3** *Let the space of  $L^{2,1}$  pointed paths  $x : [t_a, t_b] \rightarrow \mathbb{X}$  be endowed with a Gaussian integrator  $\mathcal{D}\omega_{\bar{x}, Q}(x)$ . A uniform functional on  $C'_a$  is a constraint that alters the mean but not the covariance of  $x \in X_a$  relative to  $\mathcal{D}\omega_{\bar{x}, Q}(x)$ .*

The conjugate family associated with a normal distribution of known covariance and unknown mean is again a normal distribution so the integrator associated with a uniform functional constraint is expected to be a Gaussian. Essentially this means that the unknown mean of the conditional integrator on  $X_a$  is normally distributed with respect to the marginal *and* conditional integrators on  $C_a$ .



**Definition 4.4** A Dirac integrator<sup>12</sup> is defined by

$$\mathcal{D}\delta_{\bar{c}}(c) := \mathcal{D}\omega_{\bar{c},\infty}(c) =: \delta(c - \bar{c})\mathcal{D}c \quad \text{such that} \quad \lim_{W(c') \rightarrow 0} \text{Det}[W^{1/2}]e^{-\pi W(c')} := 1. \quad (4.15)$$

**Proposition 4.1** The Dirac integrator is normalized  $\int_{C_a} \mathcal{D}\delta_{\bar{c}}(c) = 1$ , translation invariant  $\mathcal{D}\delta_{\bar{c}}(c - c_0) = \mathcal{D}\delta_{\bar{c}}(c)$ , and furnishes the functional analog of a Dirac measure

$$\int_{C_a} F_{\mu}(c) \mathcal{D}\delta_{\bar{c}}(c) = F_{\mu}(\bar{c}). \quad (4.16)$$

*Proof.* The normalization is obvious and the translation invariance follows from the translation invariance of the primitive integrator. The third relation follows trivially from definitions;

$$\begin{aligned} \int_{C_a} F_{\mu}(c) \mathcal{D}\delta_{\bar{c}}(c) &= \int_{C'_a} Z(c') d\mu(c') \\ &= \int_{C'_a} 1 d\mu(c') \\ &= F_{\mu}(\bar{c}) \end{aligned} \quad (4.17)$$

where the last line follows since  $c = \bar{c} \Rightarrow e^{\pi i \langle c', c - \bar{c} \rangle} = 1 \quad \forall c' \in C'_a$ .  $\square$

The justification for the term ‘uniform functional’ is obvious: Similar to the gamma integrator with  $\alpha = 1$  characterizing a delta functional on  $X'_a$ , the Dirac integrator characterizes a uniform functional on  $X'_a$  by  $\int_C \mathcal{D}\delta_{\bar{c}}(c) = 1$ .

More generally, the Dirac integrator yields

$$\int_{C_a} F_{\mu}(c) \mathcal{D}\delta_{\bar{c}}(M(c)) = \sum_{c_0} \frac{F_{\mu}(c_0)}{\text{Det} M'_{(c_0)}} \quad (4.18)$$

with  $M : C_a \rightarrow C_a$  a diffeomorphism and  $M(c_0 - \bar{c}) = 0$ .

To see that Dirac integrators are conjugate integrators, use (B.20) in the context of sufficient statistics. To make this concrete we stipulate that

$$G_{cx}^{-1} = D_{cx} : S_s(X_a) \rightarrow C'_a \quad ; \quad G_{xc}^{-1} = D_{xc} : C_a \rightarrow S_s(X_a)' . \quad (4.19)$$

Insofar as the sufficient statistics characterize ‘classical’ observables in a QM context, this means the constraints are only correlated with classical observables. Now taking the  $G_{cc} \rightarrow \infty$  limit gives

$$e^{-\pi Q_C(c - m_{c|x})} \longrightarrow e^{-\pi \hat{Q}(c - m_{c|x})} \delta(c - m_{c|x}) \quad (4.20)$$

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<sup>12</sup>The Dirac integrator is improper in the sense that it is a limit of a Gaussian that requires regularization to achieve a sensible normalization.

where  $\widehat{Q}(c_1, c_2) = \langle (G_{cx} D_{xx} G_{xc})^{-1} c_1, c_2 \rangle =: \langle \widehat{G}^{-1} c_1, c_2 \rangle$  and  $m_{c|x} = \bar{c} + G_{cx} D_{xx} (x - \bar{x})$ .

Clearly,  $\widehat{Q}(c_1, c_2) = Q_X(x_1, x_2)|_{S_s(X_a)}$  because of (4.19) so the exponential can be viewed as a likelihood functional

$$\Theta_{X|C}(S_s(x)|c, \cdot) = e^{-\pi Q_X(c-\bar{x})} \quad (4.21)$$

where now  $\bar{x} := (D_{xx} G_{xc})^{-1} m_{c|x}$ . It follows that

$$\Theta_{C|X}(c|x, \cdot) \sim e^{-\pi Q_X(c-\bar{x})} \delta_{\bar{c}}(c) \quad (4.22)$$

and, hence,

$$\mathcal{D}\omega_{m_{c|x}, Q_{C|X}}(c|x) \propto e^{-\pi Q_X(c-\bar{x})} \delta_{\bar{c}}(c) \mathcal{D}(x, c). \quad (4.23)$$

Since the mean is altered but not the covariance, the pair  $(\mathcal{D}\omega_{m_{c|x}, Q_{C|X}}(c), \mathcal{D}\omega_{\bar{c}, \infty}(c))$  are conjugate integrators and  $\int_{C_a} \delta(c - \bar{c}) \mathcal{D}c$  is a uniform functional constraint on  $X_a$ . Clearly the Dirac integrator implements a dynamical constraint in the sense that it can be viewed as a Lagrange multiplier term in an action functional.

An explicit example of this type of integral comes from the fixed-energy propagator on phase space often heuristically defined by

$$G_{\text{ps}}(q_b, q_a; E) = \int_{(Q, P)} \delta(H(q, p) - E) \mathcal{D}\omega^{(a, b)}(q, p) \quad (4.24)$$

where  $\mathcal{D}\omega^{(a, b)}(q, p)$  is an appropriately defined point-to-point integrator on some phase space.

Nothing is altered if we allow general action functionals. That is, we can define Dirac integrators as the limit  $S \rightarrow 0$  for exponential-type integrators. The quintessential example of this type (which, however, is outside the scope of this article) is gauge fixing in quantum field theory. Assuming well-defined functional integrals for fields, a Dirac integrator is just the Faddeev-Popov ‘trick’

$$\int_G \mathcal{D}\delta_{\bar{g}}(M(g)) = \sum_{g_o} \frac{1}{\text{Det} M'_{(g_o)}} \quad (4.25)$$

where  $M : G \rightarrow G$  and  $M(g_o - \bar{g}) = 0$ . Of course, an admissible gauge fixing condition requires a single  $g_o$ , and it is standard to average the delta functional over  $\bar{g}$  with respect to some (usually Gaussian) integrator.

The whole Faddeev-Popov procedure can be readily interpreted from a conditional integrator viewpoint

$$\begin{aligned} \int_A G(F(A)) e^{iS(A)} \mathcal{D}(F(A)) &= \int_A G(F(A)) e^{iS(A)} \text{Det} F' \mathcal{D}A \\ &= \int_{A \times C} G(c) e^{iS(A)} \text{Det} F' \delta(c - F(A)) \mathcal{D}(A, c). \end{aligned} \quad (4.26)$$

## 4.2 Quotient spaces

Let  $\tilde{\mathbb{M}}$  be a connected manifold without boundary and  $\mathbb{G} \rightarrow \mathbb{M} \xrightarrow{\Pi} \tilde{\mathbb{M}}$  a principal fiber bundle endowed with a connection. We wish to define the functional integral  $\int_{\tilde{M}_a} F_{\tilde{\mu}}(\tilde{m}) \mathcal{D}\tilde{m}$  where  $\tilde{M}_a \ni \tilde{m} : [t_a, t_b] \rightarrow \tilde{\mathbb{M}}$ . The problem is that the base space is complicated in general: It may be very difficult or impossible to *directly* define an integrator on  $\tilde{\mathbb{M}}$ .

On the other hand, the covering space is usually easier to handle, and we assume that we can define an integrator so that the integral  $\int_{M_a} F_{\mu}(m) \mathcal{D}m$  is well-defined. We also assume that  $F_{\mu}(m)(t)$  furnishes a representation of  $\mathbb{G}$  and is equivariant so that

$$F_{\mu}(m)(t) \cdot g = \rho(g^{-1}) F_{\mu}(m)(t) \quad (4.27)$$

where  $\rho$  is a possibly non-faithful representation of  $\mathbb{G}$ .

Now, in the CDM scheme expressions like  $m(t)$  are shorthand for a parametrized curve:  $m(b)(t) = m_a \cdot \Sigma(t, b)$  where  $b \in B_a$  and  $\Sigma(t, b) : \mathbb{M} \rightarrow \mathbb{M}$  is a global transformation such that  $\Sigma(t_a, \cdot) = m_a$ . This parametrization allows integrals over the generally non-Banach space  $M_a$  to be expressed as well-defined integrals over  $B_a$ .

The first point to make is that the parametrization is gleaned from the local structure of the bundle  $\mathbb{U}_i \times \mathbb{G}$  where  $\mathbb{U}_i \subset \tilde{\mathbb{M}}$  and a local trivialization is given so that  $m(b)(t) = (\tilde{m}(b), g(b))(t)$ . The parametrization for the first component is dictated by the manifold structure of  $\tilde{\mathbb{M}}$ . The parametrization for the second component is fixed by requiring parallel transport of  $m(t)$  — since this will restrict paths to  $\tilde{\mathbb{M}}$  if that is where they start. Consider an open set  $\mathbb{U}_i \subset \tilde{\mathbb{M}}$ , and let  $A_i$  denote the local gauge potential relative to the *canonical* local section  $s_i : \tilde{\mathbb{M}} \rightarrow \mathbb{M}$ . The equation for parallel transport,

$$dg_i(t) = -g_i(t) A_i(\tilde{m}(t)) dt \quad (4.28)$$

clearly indicates the conditional relation between  $\tilde{m}$  and  $g$ , and it emphasizes the interplay between constraints and conditionals.

To attack this problem we need to be more explicit about the parametrization of  $M_a$ .

**Definition 4.5** Let  $\{\omega_i = 0\}$  where  $\omega_i \in \Lambda T^* M_a$  and  $i \in \{1, \dots, r\}$  be an exterior differential system with integral manifold  $(B, M_a)$ . This system defines a parametrization  $P : B \rightarrow M_a$  by

$$P^* \omega_i = 0 \quad \forall i. \quad (4.29)$$

In particular, if  $B = X \times Y$  and  $i = 2$ , the parametrization can be written locally on  $\mathbb{M}$  as

$$\begin{cases} d\tilde{m}(x)(t) = \mathbf{X}_{(a)}(m(x)(t)) dx(t)^a & \tilde{m}(t_a) = \tilde{m}_a \\ dg(y)(t) = \mathbf{Y}_{(b)}(m(y)(t)) dy(t)^b & g(t_a) = g_a \end{cases} \quad (4.30)$$

where  $a \in \{1, \dots, m_a\}$ ,  $b \in \{1, \dots, m_b\}$ ,  $m_a + m_b \leq \dim \mathbb{M}$ , and the set  $\{\mathbf{X}_{(a)}, \mathbf{Y}_{(b)}\}$  generates a vector sub-bundle  $\mathbb{V} \subseteq T\mathbb{M}$  of the tangent bundle. The solution of (4.30) will be denoted  $m(x, y)(t) = m_a \cdot \Sigma(t, x, y)$  where  $\Sigma(t, x, y) : \mathbb{M} \rightarrow \mathbb{M}$  is a global transformation such that  $\Sigma(t_a, \cdot, \cdot) = \text{Id}$ .

The parallel transport equation implies  $dg(t) \sim A \cdot d\tilde{m}(t)$  which implicitly encodes the constraint through (4.30). This particular parametrization yields (within  $\mathbb{U}_i$ )

$$m(x)(t) = (\tilde{m}(x), g(\tilde{m}(x)))(t) = (\tilde{m}_a \cdot \tilde{\Sigma}(t, x), g_a \cdot \mathcal{P}e^{-\int_{t_a}^t A \cdot \dot{\tilde{m}} dt}) \quad (4.31)$$

where  $\tilde{\Sigma} = \Pi(\Sigma)$ . On the other hand, since  $m$  is a horizontal lift,

$$m(x)(t) = s_i(\tilde{m}(x)(t)) \cdot g(\tilde{m}(x))(t). \quad (4.32)$$

It is clear that the restriction expressed by (4.28) implies  $g(\tilde{m}(x))(t) \in \mathbb{G} \forall \{t, x\}$ , and  $m(x) : [t_a, t_b] \rightarrow \mathbb{M}(m_a)$  where  $\mathbb{H}(m_a) \rightarrow \mathbb{M}(m_a) \xrightarrow{\Pi} \tilde{\mathbb{M}}$  is the holonomy bundle.

We have learned that, given  $\tilde{m}(x)$  and a local trivialization, the constraint  $\delta(M(g)) = \delta(\dot{g} - g A \cdot \dot{\tilde{m}})$  will lead to a path confined to an open neighborhood of a section that is isomorphic to the base space. And since the constraint only shifts the path along fibers, we expect it to be represented by a uniform functional which implies a Dirac integrator  $\mathcal{D}\delta_{ge}(M(g))$  should be used. Moreover,  $M$  just effects translation on  $G_a$  so the functional determinant of  $M'$  is trivial and the zero locus coincides with the holonomy group  $\mathbb{H}(m_a)$ .

So, for a Gaussian integrator,

$$\begin{aligned} \int_{\tilde{M}_a} F_{\tilde{\mu}}(\tilde{m})(t) \mathcal{D}\omega(\tilde{m}) &:= \int_{X_a} \int_{G_a} F_{\mu}(m(x))(t) \delta(M(g)) \mathcal{D}g \mathcal{D}\omega_{\tilde{x}, Q}(x) \\ &= \int_{\mathbb{H}(m_a)} \int_{X_a} F_{\mu}(s_i(\tilde{m}(x))(t) \cdot h) \mathcal{D}\omega_{\tilde{x}, Q}(x) dh \\ &= \int_{\mathbb{H}(m_a)} \int_{X_a} \rho(h^{-1}) F_{\mu}(m(x))(t) \mathcal{D}\omega_{\tilde{x}, Q}(x) dh \end{aligned} \quad (4.33)$$

where  $m(x)(t) \in \mathbb{M}(m_a)$ .

In particular, the point-to-point propagator on  $\tilde{\mathbb{M}}$  obtains for the familiar choice  $F_{\mu}(m(x))(t_b) = \delta(m(x)(t_b), m_b)$ ;

$$\begin{aligned} \tilde{K}(\tilde{m}_a, \tilde{m}_b) &:= \int_{\tilde{M}_a} \delta(\tilde{m}(t_b), \tilde{m}_b) \mathcal{D}\omega(\tilde{m}) \\ &= \int_{\mathbb{H}(m_a)} \rho(h^{-1}) \int_{X_a} \delta(m(x)(t_b), m_b) \mathcal{D}\omega_{\tilde{x}, Q}(x) dh \\ &=: \int_{\mathbb{H}(m_a)} \rho(h^{-1}) K_{h_b}(\tilde{m}_a, \tilde{m}_b) dh \end{aligned} \quad (4.34)$$

where the propagator  $K_{h_b}(\tilde{m}_a, \tilde{m}_b)$  is the point-to-point propagator on  $\mathbb{M}(m_a)$  — associated with the homotopy class of paths indexed by  $h_b$  — pulled back to  $\tilde{\mathbb{M}}$ .<sup>13</sup>

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<sup>13</sup>It is legitimate to write  $K_{h_b}(\tilde{m}_a, \tilde{m}_b)$  instead of  $K(m_a, m_b) = K(s_i(\tilde{m}_a), s_i(\tilde{m}_b) \cdot h_b)$  because the horizontal lifting does not depend on the trivialization and, hence, the particular *canonical* section  $s_i$ . So we are free to choose the trivial section. Recall that  $K(m_a, m_b)$  represents the point-to-point propagator, so its evaluation may include a sum over relevant paths.

In most physical systems, the integrator  $\mathcal{D}\omega_{\bar{x},Q}(x)$  is invariant under the restricted holonomy group  $\mathbb{H}_{(m_a)}^0$  acting on  $X_a$ . Then  $K(m_a, m_b \cdot h^0) = K(m_a, m_b)$  where  $h^0 \in \mathbb{H}_{(m_a)}^0$  and the propagator on the base space reduces to the well-known result (for  $\chi : \mathbb{G} \rightarrow \mathbb{C}$ )

$$\tilde{K}(\widetilde{m}_a, \widetilde{m}_b) = \sum_{\Lambda_{m_a}} \tilde{K}^{\Lambda_{m_a}}(\widetilde{m}_a, \widetilde{m}_b) = \sum_{\Lambda_{m_a}} \sum_{g \in \mathbb{G}} d_{\Lambda_{m_a}} \chi^{\Lambda_{m_a}}(g) K_{[h_b]}(\widetilde{m}_a, \widetilde{m}_b) \quad (4.35)$$

where  $\Lambda_{m_a}$  labels the representation of the monodromy group  $\mathbb{G} = \mathbb{H}_{(m_a)}/\mathbb{H}_{(m_a)}^0$  at the point  $m_a$  and  $d_{\Lambda_{m_a}}$  its multiplicity.

On the other hand, since the free point-to-point propagator on the group manifold is known to be exact, then by (4.12) we get a trace formula for point-to-point transitions

$$\sum_{\bar{x}} \frac{e^{\pi i B(\bar{x}(t_b))}}{\sqrt{\det[i\mathbf{G}(t_b, t_b)]}} = \sum_{\Lambda_{m_a}} \sum_{g \in \mathbb{G}} d_{\Lambda_{m_a}} \chi^{\Lambda_{m_a}}(g) K_{[h_b]}(\widetilde{m}_a, \widetilde{m}_b) . \quad (4.36)$$

### 4.3 Discontinuous spaces

#### 4.3.1 Boundaries

We wish to define an integrator for a space of pointed paths  $M_a^\partial$  with  $m : [t_a, t_b] \rightarrow \mathbb{M}$  and  $\partial\mathbb{M} \neq \emptyset$  sufficiently regular. Experience from the previous sections indicates that we should take  $B_a = X_a \times T_a$  and consider a product manifold  $\mathbb{N} = \mathbb{M} \times \mathbb{R}_+$ .

The task is to make sense of an integral of the form  $\int_{M_a^\partial} F_\mu(m) \mathcal{D}m$ . So, for a Gaussian integrator on  $X_a$ , take a gamma integrator on  $T_a$  to account for the boundary constraint. The parametrization of  $n := (m, \tau) : [t_a, t_b] \rightarrow \mathbb{N}$  can be written

$$\begin{cases} dm(x)(t) = \mathbf{X}_{(a)}(m(x)(t))dx(t)^a & m(x)(t_a) = m_a \\ d\tau(t) = \mathbf{Y}(\tau(t))dt & \tau(t_a) = \tau_a = 0 \end{cases} , \quad (4.37)$$

and the integral defined by

$$\begin{aligned} \int_{M_a^\partial} F_\mu(m) \mathcal{D}m &:= \int_{B_a} F_\mu(n(x|\tau)) \Theta_{X|T}(x|\tau, \cdot) \mathcal{D}\Theta_{X|T, Z_{X|T}} x | \tau \\ &=: \int_{B_a} \tilde{F}_\mu(n(x, \tau), \cdot) \mathcal{D}\omega_{\bar{x},Q}(x) \mathcal{D}\gamma_{\alpha, \beta', \tau_\partial} \\ &= \int_{T_a} \left\langle \tilde{F}_\mu(n(x, \tau)) \right\rangle_{\bar{x}, Q} \mathcal{D}\gamma_{\alpha, \beta', \tau_\partial} \\ &=: \int_{T_a} \tilde{H}_\mu(n(S_s(x), \tau)) \mathcal{D}\gamma_{\alpha, \beta', \tau_\partial} . \end{aligned} \quad (4.38)$$

It remains to infer the nature of  $\tilde{H}_\mu(n(S_s(x), \tau))$  and the associated integrator parameters.

From the variational principle, we learned that the constraints impose transversality conditions on *critical* paths. We also learned that the physically meaningful path parameter is to be associated with  $\tau$  and not  $t$ . So the plan is to search for a conditional integrator for which<sup>14</sup>

$$\Theta_{T|X}(\tau|x, \cdot) \propto \Theta_{S_s(X)|T}(S_s(x)(\tau), \cdot) \Theta_T(\tau, \cdot) \quad (4.39)$$

where  $S_s(X)$  is determined by *mean* paths. We replace critical with mean paths for the sufficient statistic because the quantum analog of the variational principle is

$$\frac{\delta \Gamma(\bar{x})}{\delta \bar{x}(t)} \sim x'(t) \quad (4.40)$$

with  $\Gamma$  defined in (B.9).

To simplify matters, only the two limiting cases of transversal intersection and fixed energy discussed in §2 will be considered. Recall that these cases correspond to point-to-boundary and point-to-point paths respectively. Let  $\bar{n} = (\bar{m}, \bar{\tau})$  represent a mean path, and define the “first exit” time  $t^\perp$  by  $\bar{n}(t^\perp) := (m_B, \tau^\perp)$  where  $m(t^\perp) = m_B \in \partial\mathbb{M}$  and  $\tau^\perp := \tau(t^\perp)$ . Recall the mean path  $\bar{m}$  is also a critical path of  $Q(m(x))$  since  $Q$  is quadratic, and there may be more than one critical path.

The functional integral for a functional  $F_\mu^\partial$  that takes its values on  $\partial\mathbb{M}$  is defined to be

$$\Phi_\partial(m_a) := \int_{B_a} F_\mu^\partial(m(x)(\tau^\perp)) \mathcal{D}\omega_{\bar{m}^\perp, Q}(x) \mathcal{D}\gamma_{0,0,\infty}(\tau) \quad (4.41)$$

Likewise, for a functional  $F_\mu^{\setminus\partial}$  that takes its values in  $\mathbb{M} \setminus \partial\mathbb{M}$ ,

$$\Phi_{\setminus\partial}(m_a) := \int_{B_a} F_\mu^{\setminus\partial}(m(x)(\tau)) \mathcal{D}\omega_{\bar{m}^E, Q}(x) \mathcal{D}\gamma_{1,0,\tau_\partial}(\tau) \quad (4.42)$$

where  $\overline{\tau_\partial} := \tau^\perp$ . These definitions hold for  $Q \rightarrow S$ , but the mean paths are no longer critical. Of course, the suitability of these definitions rests on their ability to reproduce known results (see e.g. [6]).

Like the quotient space example, the two integrals simplify considerably for propagators. For the point-to-boundary propagator<sup>15</sup>,

$$\begin{aligned} K_\partial(m_a, m_B) &= \mathcal{N}(m_a) \int_{B_a} \delta(m(x)(\tau^\perp), m_B) \mathcal{D}\omega_{\bar{m}^\perp, Q}(x) \mathcal{D}\gamma_{0,0,\infty}(\tau) \\ &= \mathcal{N}(m_a) \sum_{\bar{m}^\perp} \int_{T_a} \frac{1}{\sqrt{\det i \mathbf{G}(\tau^\perp, \tau^\perp)}} \mathcal{D}\gamma_{0,-\pi i B(\bar{m}^\perp), \infty}(\tau) \\ &= \mathcal{N}(m_a) \sum_{\bar{m}^\perp} \int_{\mathbb{R}_+} \frac{e^{\pi i B(\bar{m}^\perp)}}{\sqrt{\det i \mathbf{G}(\tau^\perp, \tau^\perp)}} d(\ln \tau^\perp) \end{aligned} \quad (4.43)$$

<sup>14</sup>The same idea was implicit in the quotient space analysis. There,  $\Theta_{X|G}$  was determined by (4.28).

<sup>15</sup>The need for the normalization constant  $\mathcal{N}(m_a)$  can be established from dimensional analysis.

where the normalization constant  $\mathcal{N}(\mathbf{m}_a) = \int_{\partial} K_{\partial}(\mathbf{m}_a, \mathbf{m}_B) d\mathbf{m}_B$  and the end-point is  $\mathbf{m}_B := m(\bar{x}(\tau^{\perp}))$ . For example, if  $\mathbb{M} \subset \mathbb{R}^n$ , then the boundary term goes like  $B(\bar{m}^{\perp}) \sim |\mathbf{m}_B - \mathbf{m}_a|^2 / \tau^{\perp}$ . If there is more than one critical (or mean) path, care must be taken to split the integral over the boundary into regions associated with a particular critical path.

For the point-to-point propagator,

$$\begin{aligned} K(\mathbf{m}_a, \mathbf{m}_b) &= \int_{B_a} \delta(m(x)(\tau_b), \mathbf{m}_b) \mathcal{D}\omega_{\bar{m}^E, Q}(x) \mathcal{D}\gamma_{1,0,\tau_{\partial}}(\tau) \\ &= \sum_{\bar{m}^E} \int_{T_a} \frac{1}{\sqrt{\det i \mathbf{G}(\tau_b, \tau_b)}} \mathcal{D}\gamma_{1,-\pi i B(\bar{m}^E), \tau_{\partial}}(\tau) \\ &= \sum_{\bar{m}^E} \int_0^{\tau^{\perp}} \frac{e^{\pi i B(\bar{m}^E)}}{\sqrt{\det i \mathbf{G}(\tau_b, \tau_b)}} d\tau_b \end{aligned} \quad (4.44)$$

where  $\mathbf{m}_b := m(\bar{x}(\tau_b))$ . Incidentally, for  $\mathbb{M} \subset \mathbb{R}^n$ , it is not hard to show using the explicit expression for  $B(\bar{m}^E)$  and Proposition 3.1 from [6] that the normal derivative of  $K$  evaluated on the boundary is just  $K_{\partial}$  as it should be. In essence,

$$\nabla_{\mathbf{m}_a} \int_{T_a} \mathcal{D}\gamma_{1,-\pi i B(\bar{m}), \tau_{\partial}}(\tau) \Big|_{\partial} \cdot \hat{\mathbf{n}} = \mathcal{N}(\mathbf{m}_a) \int_{T_a} \mathcal{D}\gamma_{0,-\pi i B(\bar{m}), \infty}(\tau) . \quad (4.45)$$

These point-to-point and point-to-boundary propagators are *restricted* in the sense that the paths are not allowed to penetrate the boundary. However, there are cases of interest when the boundary represents a discontinuity, and the paths are defined on both sides of the boundary.

### 4.3.2 Segmented configuration space

Often the target space of paths will have codimension-1 submanifolds that induce a decomposition of path space  $\bigoplus_i X_a^{(i)} = X_a$  in the sense that each  $X_a^{(i)}$  has its own integrator. Such is the case, for example, when a Gaussian integrator is defined in terms of an action functional and the potential in the action is discontinuous. More explicitly,  $x_a^{(i)} \in X_a^{(i)}$  is the pointed path  $x_a^{(i)} : [t_a, t_b] \rightarrow (\mathbb{M}^{(i)}, \mathbf{m}_a^{(i)})$  where  $\mathbb{M} = \bigcup_i \mathbb{M}^{(i)}$  such that the intersection  $\mathbb{M}^{(i)} \cap \mathbb{M}^{(j)} = \partial \mathbb{M}^{(ij)}$  is a submanifold of codimension-1.

The objects of interest in this case are the propagators from the previous subsection. Since it is known that the propagators are kernels for certain differential operators, Green's theorem provides a convenient starting point for the analysis.

For an operator  $L$  defined in a bounded open region  $\mathbb{U} \subset \mathbb{M}$  acting on complex scalar functions from an appropriate function space,

$$\int_{\mathbb{U}} (L\phi) \bar{\varphi} - \int_{\mathbb{U}} \phi \overline{(L^* \varphi)} = \int_{\partial \mathbb{U}} B(\phi, \bar{\varphi}) \quad (4.46)$$

where  $L^*$  is the formal adjoint of  $L$ , and the functional form of  $B$  is determined from Stoke's theorem and the particular boundary conditions associated with the function space.

In particular, let  $\mathbb{U}_1 = \mathbb{U}^{(1)} \cup \mathbb{U}^{(2)}$  be a bounded open region in  $\mathbb{R}^3$  with one surface  $\mathbb{S} = \mathbb{U}^{(1)} \cap \mathbb{U}^{(2)}$  of discontinuity. Choose  $\varphi$  to be the Green's function of  $\nabla^*$  in  $\mathbb{U}_1$  with vanishing Dirichlet boundary conditions on  $\partial\mathbb{U}_1$  and  $\phi$  the Green's function of  $\nabla$  in  $\mathbb{U}^{(1)}$  with vanishing Dirichlet conditions on  $\mathbb{S}$ . Then the theorem gives the Green's function  $\bar{\varphi}$  of  $\nabla$  in  $\mathbb{U}^{(1)}$  with *non-vanishing* boundary conditions on  $\mathbb{S}$

$$\bar{\varphi} = \phi + \int_{\mathbb{S}} \bar{\varphi} \nabla \phi \cdot d\sigma \quad (4.47)$$

and in  $\mathbb{U}_1 \setminus \mathbb{U}^{(1)}$

$$\bar{\varphi} = \int_{\mathbb{S}} \bar{\varphi} \nabla \phi \cdot d\sigma . \quad (4.48)$$

This theorem has a simple interpretation in terms of functional integral representations of propagators: For bounded regions that allow paths to penetrate the boundary (e.g. the discontinuity we are dealing with) the total point-to-point propagator includes the *restricted* point-to-point propagator, which does not allow paths to leave the region, plus the potential on the bounding surface induced by all sources accessible to paths that are allowed to leave the region. This prescription is equivalent to the “path decomposition technique” used in [7],[8],[9].

In other words, Green's theorem can be used to partition the space of paths taking their values in a segmented configuration space into restricted and unrestricted sets. This is useful because the paths in the partitioned sets have particularly convenient boundary conditions and their associated propagators are relatively easy to calculate.

Let  $K_{\mathbb{U}^{(i)}}^{(D)}$  be the restricted point-to-point propagators with Dirichlet boundary conditions on  $\mathbb{S}$ , and  $K_{\partial^{(i)}}^{(D)}$  the restricted point-to-boundary homogenous propagators for their respective regions  $\mathbb{U}^{(i)}$ . These are the propagators derived in the previous subsection. According to Green's theorem, the *unrestricted* point-to-point propagator from  $m_a^{(i)}$  to  $m_b^{(j)}$  in  $\mathbb{U}_1$  with one surface of discontinuity can be written

$$K_{\mathbb{U}_1}^{(D)}(m_a^{(i)}, m_b^{(j)}) := \delta_{ij} K_{\mathbb{U}^{(j)}}^{(D)}(m_a^{(i)}, m_b^{(j)}) + \int_{\mathbb{S}} K_{\partial^{(i)}}^{(D)}(m_a^{(i)}, \sigma) K_{\mathbb{U}_1}^{(D)}(\sigma, m_b^{(j)}) d\sigma \quad (4.49)$$

where  $m^{(i)} \in \mathbb{U}^{(i)}$  and  $\sigma \in \mathbb{S}$ . Intuitively, the unrestricted point-to-point propagator within a bounded region  $\mathbb{U}^{(i)}$  is implicitly determined by the restricted point-to-point and point-to-boundary propagators in  $\mathbb{U}^{(i)}$ . Similarly, the point-to-point propagator from  $\mathbb{U}^{(i)}$  to  $\mathbb{U}^{(j)}$  is implicitly determined by the restricted point-to-boundary propagator in  $\mathbb{U}^{(i)}$ . Note that  $K_{\mathbb{U}_1}^{(D)}$  has non-trivial boundary conditions on  $\mathbb{S}$ , but it still satisfies Dirichlet boundary conditions on  $\partial\mathbb{U}$ .

Equation (4.49) is a familiar expression, and it is often solved by way of a Born approximation. However, the path decomposition idea along with the fact that  $K_{\mathbb{U}_1}^{(D)}$  *inside the integral is evaluated on the surface*, suggests<sup>16</sup> that we replace  $K_{\mathbb{U}_1}^{(D)}$  (inside

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<sup>16</sup>This argument is admittedly highly heuristic.



the integral) with  $\tilde{K}_{\mathbb{U}_0}^{(D)}$  where

$$\tilde{K}_{\mathbb{U}_0}^{(D)}(\sigma, \mathbf{m}_b^{(j)}) := \begin{cases} r_{(\neg i)}(\sigma) K_{\mathbb{U}_0|_i}^{(D)}(\mathbf{m}_{a'}^{(i)}, \mathbf{m}_b^{(i)})|_{\mathbf{m}_{a'}^{(i)}=\sigma} & \text{if } i = j \\ t_{(\neg j)}(\sigma) K_{\mathbb{U}_0|_j}^{(D)}(\mathbf{m}_{a'}^{(j)}, \mathbf{m}_b^{(j)})|_{\mathbf{m}_{a'}^{(j)}=\sigma} & \text{if } i \neq j \end{cases}, \quad (4.50)$$

where  $K_{\mathbb{U}_0|_j}^{(D)}$  is the unrestricted propagator evaluated in  $\mathbb{U}^{(j)}$ , and

$$\begin{aligned} r_{(\neg i)}(\sigma) &= \int_{\mathbb{S}} K_{\mathbb{U}_0|_{\neg i}}^{(D)}(\sigma, \sigma') K_{\partial(\neg i)}^{(D)}(\sigma', \sigma) d\sigma' \\ t_{(\neg j)}(\sigma) &= \int_{\mathbb{S}} K_{\mathbb{U}_0|_{\neg j}}^{(D)}(\sigma, \sigma') K_{\partial(\neg j)}^{(D)}(\sigma', \sigma) d\sigma'. \end{aligned} \quad (4.51)$$

are evaluated in the region on the other side  $(\neg j)$  of  $\mathbf{m}_b^{(j)}$ . According to the path decomposition picture,  $r(\sigma)$  and  $t(\sigma)$  measure the (probability amplitude) contribution of pointed loops based at  $\sigma$  that lie on either side of  $\mathbb{S}$  and so  $|r|^2 + |t|^2 = 1$ .

Usually it is much simpler to find  $\tilde{K}_{\mathbb{U}_0}^{(D)}$  than to iterate (4.49). For example, take the fixed energy propagator in  $\mathbb{R}$  with a step potential  $V(x) = V_0\theta(x)$ . Let  $k_0$  and  $k_{V_0}$  be the wave vectors to the left and right of  $x = 0$  respectively. Then it is easy to see from the surface integral that  $t \sim 2\sqrt{k_0 k_{V_0}}/(k_0 + k_{V_0})$  since this must be symmetric under  $k_0 \leftrightarrow k_{V_0}$ ; and, hence,  $r \sim (k_0 - k_{V_0})/(k_0 + k_{V_0})$  from the normalization condition.

Now let  $\mathbb{U}$  be divided into three regions. We can use the results for a single surface of discontinuity by covering  $\mathbb{U}$  with two overlapping copies of  $\mathbb{U}_1$ . That is, each region contains only one surface of discontinuity. There are now six relevant propagators with appropriate boundary conditions. Their nature depends on whether or not the boundaries intersect  $\partial\mathbb{U}$ . Since we require Dirichlet boundary conditions for point-to-point transitions and  $K_{\mathbb{U}_1}^{(D)}$  can only propagate across a single discontinuity, care must be taken to use the appropriate  $\mathbb{U}_1$  for any given transition.

To simplify, specialize to a planar geometry, and order the regions  $\{1, 2, 3\}$ . There will be two classes of propagators; half-space-type in regions 1 and 3 defined by  $\overline{\mathbb{U}_1 \cap \mathbb{U}_{1'}}$ , and unit-strip-type in region 2 defined by  $\mathbb{U}_1 \cap \mathbb{U}_{1'}$  where  $\mathbb{U}_1 = \mathbb{U}^{(1)} \cup \mathbb{U}^{(2)}$  and  $\mathbb{U}_{1'} = \mathbb{U}^{(2)} \cup \mathbb{U}^{(3)}$ . Choose the partition  $\mathbb{U} = \mathbb{U}_1 \cup \mathbb{U}^{(i)}$  so that  $\mathbb{U}_1 \equiv \mathbb{U}^{(j)}$  contains the initial point  $x_{a'}^{(j)}$ . By combining the single surface result appropriately, the point-to-point propagator for two surfaces of discontinuity can be written

$$\begin{aligned} K_{\mathbb{U}_2}^{(D)}(\mathbf{x}_a^{(i)}, \mathbf{x}_{a'}^{(j)}) &:= \delta_{ij} K_{\mathbb{U}^{(j)}}^{(D)}(\mathbf{x}_a^{(i)}, \mathbf{x}_{a'}^{(j)}) \\ &+ \int_{\mathbb{S}_2} K_{\partial^{(j)}}^{(D)}(\mathbf{x}_a^{(i)}, \sigma^{(j)}) \tilde{K}_{\mathbb{U}_1}^{(D)}(\sigma^{(j)}, \mathbf{x}_{a'}^{(j)}) d\sigma^{(j)} \end{aligned} \quad (4.52)$$

where  $K_{\mathbb{U}^{(j)}}^{(D)}$  is the restricted point-to-point propagator and  $K_{\partial^{(j)}}^{(D)}$  the restricted point-to-boundary propagator derived from (4.49).

It is important to remember that the propagator depends on *all* critical paths. For regions bounded by two planes, there is obviously a sum over all ‘bounces’ within

the bounded region<sup>17</sup>. These bounce transitions are encoded in the  $K_{\partial(j)}(\mathbf{x}_a^{(i)}, \sigma^{(j)})$  propagator. Hence, poles of the convolution of relevant bounce propagators yield spectral information for their corresponding regions.

In general then, the point-to-point propagator for  $\mathbb{U} = \bigcup_{i=1}^{n+1} \mathbb{U}^{(i)}$  is determined recursively from  $K_{\mathbb{U}_0}^{(D)}$ , which is the standard unrestricted point-to-point elementary kernel in the region  $U^{(j)}$  with vanishing Dirichlet boundary conditions on  $\partial\mathbb{U}$ , and

$$\begin{aligned} K_{\mathbb{U}_n}^{(D)}(\mathbf{x}_a^{(i)}, \mathbf{x}_{a'}^{(j)}) &:= \delta_{ij} K_{\mathbb{U}^{(j)}}^{(D)}(\mathbf{x}_a^{(i)}, \mathbf{x}_{a'}^{(j)}) \\ &+ \int_{\mathbb{S}_{(n)}} K_{\partial(j)}^{(D)}(\mathbf{x}_a^{(i)}, \sigma^{(j)}) \tilde{K}_{\mathbb{U}_{n-1}}^{(D)}(\sigma^{(j)}, \mathbf{x}_{a'}^{(j)}) d\sigma^{(j)} \end{aligned} \quad (4.53)$$

where the region  $\mathbb{U}_{n-1}$  is chosen to contain the initial point.

There are cases when the iteration of (4.49) can be summed explicitly. The method (see e.g. [12]) essentially boils down to finding a Poisson integrator that is valid everywhere in  $X_a$ . To see this, let  $S(x) = Q(x) + V(x)$  describe some general (action) functional, and suppose the propagator  $K_Q$  has been found in each  $\mathbb{U}^{(i)}$ . Define  $\tilde{V} = K_Q \circ V \circ K_Q$  by its evaluation on ordered graphs in  $\mathbb{M}$ , i.e. under the linear maps  $L_n : T_a \rightarrow \overline{\mathbb{R}}_+^n$  we have  $m(x)(\tau) \mapsto (m(x(\tau_1)), \dots, m(x(\tau_n))) =: (\mathbf{m}_1, \dots, \mathbf{m}_n) \in \mathbb{M}^n$ . Then

$$\tilde{V}(m(x(\tau))) \rightarrow K_Q(\mathbf{m}_b, \mathbf{m}_k) V(\mathbf{m}_k) K_Q(\mathbf{m}_k, \mathbf{m}_{k-1}), \dots, V(\mathbf{m}_1) K_Q(\mathbf{m}_1, \mathbf{m}_a) \quad (4.54)$$

where  $\mathbf{m}_a \leq \mathbf{m}_1, \leq \dots \leq \mathbf{m}_b$  are the time-ordered graph nodes. A time-slicing analysis ([12]) when  $\mathbb{M} = \mathbb{R}^m$  shows the propagator has the same form given by

$$K_S(\mathbf{m}_a, \mathbf{m}_b) := K_S(\tau_a, \tau_b) = \langle \tilde{V} \rangle_{\lambda, \tau_b} \quad (4.55)$$

where  $\langle \cdot \rangle_{\lambda, \tau_b}$  is defined in subsection B.0.4.

But the time-slicing analysis is only straightforward in  $\mathbb{R}^m$ ; otherwise there are well-known pitfalls. However, the target manifold independence of the right-hand side of (4.55) suggests that the definition is correct for any manifold  $\mathbb{M}$ . The point is, (4.55) is more general than a perturbation expansion: it is defined in the function space rather than on the target manifold  $\mathbb{M}$ . Consequently it offers potentially new insight and calculational techniques.

In particular, an obvious conjecture is that  $K_S(\tau_a, \tau_b)$  can be represented by

$$K_S(\tau_a, \tau_b) \sim \int_{\mathcal{C}} \int_{T_a} \left[ \tilde{V}(\tau) \mathcal{D}_{\gamma_{\alpha, \beta', \tau_b}}(\tau) \right] d\alpha \quad (4.56)$$

for an appropriate contour  $\mathcal{C} \subset \mathbb{C}$ . This is a substantial generalization of the Poisson average because now the target space is  $\mathbb{C}$  rather than  $\mathbb{R}_+$ . Not only do we get phase

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<sup>17</sup>To the extent that the boundaries are exactly parallel and/or the planes extend to infinity, this gives an infinite sum which can be written analytically in the usual way as an inverse propagator.

information not carried by the Poisson integrators, but we don't have to restrict to  $\langle \beta', \tau \rangle = \lambda \langle \text{Id}', \tau \rangle$ .

Here again is the thematic idea connecting propagators to summation over an integrator family. According to the conjecture, the analytic structure of  $K_S(m_a, m_b)$  could be encoded in the integral defined by  $K_S(\tau_a, \tau_b; \alpha) := \Gamma(1 - \alpha) \int_{T_a} \tilde{V}(\tau) \mathcal{D}\gamma_{\alpha, \beta', \tau_b}(\tau)$ . Indeed, if  $\tilde{V}(\tau) = V \in \mathbb{C}$ ,  $\langle \beta', \tau \rangle = \lambda \langle \text{Id}', \tau \rangle$ ,  $|\lambda \tau_b| > 1$ , and  $\Re(\lambda \tau_b) > 0$ , we get

$$\begin{aligned}
K_S(\tau_a, \tau_b) &= \int_{\mathcal{C}} \frac{\Gamma(1 - \alpha)}{2\pi i} \int_{T_a} [V \mathcal{D}\gamma_{\alpha, \beta', \tau_b}(\tau)] d\alpha \\
&= V \int_{\mathcal{C}} \frac{\Gamma(1 - \alpha)}{2\pi i} \gamma(\alpha, \lambda \tau_b) d\alpha \\
&= V \left[ \sum_{n=-\infty}^0 \frac{\Gamma(1 + n)(-1)^{-n}}{n!} P(-n, \lambda \tau_b) \right] \\
&= V \sum_{n=0}^{\infty} (-1)^n P(n, \lambda \tau_b) \\
&= V.
\end{aligned} \tag{4.57}$$

where the contour encloses the poles of  $\gamma(\alpha, \tau_b)$  on the negative real axis. And this agrees with definition (B.48) of the Poisson average  $\langle V \rangle_{\lambda, \tau_b}$ .

Parenthetically, it is interesting to note that the case  $|\lambda \tau_b| < 1$  can be handled in the same way by using the inverse gamma integrator, defined in the obvious way, to define an inverse Poisson propagator. Like the gamma integrator, the inverse gamma integrator is also a conjugate family for a Gaussian of known mean so everything hangs together.

## 5 Prime examples

Our goal in this section is to give functional integral representations of the prime counting function and some of its relatives. We formulate the counting functions in the spirit of quantum mechanical expectation values in the sense that they will represent the sum over constrained ‘paths’ with certain attributes. Specifically, the paths are conjectured to follow gamma rather than Gaussian statistics.

### 5.1 Prime counting

Postulate that the prime counting function is the expectation of a gamma process with unknown scaling parameter due to the constraint associated with counting only primes. The functional integral that enforces the constraint must be a gamma integral because the conjugate prior of a gamma distribution with unknown scaling parameter is again a gamma distribution. Therefore, according to the general construction, the constrained functional integral that represents the expectation value

can be written as a constrained functional which is integrable with respect to two marginal gamma integrators. Similar to the QM point-to-point free propagator example, we put  $i\langle\beta(\tau_b)', c\rangle = i(\tau_b - \lambda(\tau_b)) \cdot \bar{c}$ . Then let us define the average prime counting function to be

$$\begin{aligned}\overline{\pi_1(\tau_b)} &:= \text{tr}_\alpha \langle \tau \rangle_{\tau_b} := \text{tr}_\alpha \int_{T_a \times C_a} \tau \mathcal{D}\gamma_{1, i\beta(\tau_b)', \infty}(c) \mathcal{D}\gamma_{\alpha-2, -\text{Id}', \tau_\partial}(\tau) \\ &= \text{tr}_\alpha \int_{T_a \times C_a} \mathcal{D}\gamma_{\alpha-1, -\text{Id}', \tau_b}(\tau) \mathcal{D}\gamma_{1, i\beta(\tau_b)', \infty}(c)\end{aligned}\quad (5.1)$$

where  $\langle\beta(\tau_b)', c\rangle = \langle\langle\delta_{t_b}, (\tau_\partial - \lambda(\tau_\partial))\rangle', c\rangle \in \mathbb{R}_+$  such that  $\tau_\partial(t_b) = \tau_b$  and  $\lambda(\tau_b)$  represents an unknown possibly non-homogenous scaling factor, and the  $\alpha$ -trace is defined below.

Using the definition of a delta functional, and recalling the treatment of  $K(\tau_a, \tau_b; \alpha)$  from the previous section yields

$$\begin{aligned}\overline{\pi_1(\tau_b)} &= \text{tr}_\alpha \int_{C_a} (-1)^{\alpha-1} \gamma(\alpha-1, -\tau_b) \mathcal{D}\gamma_{1, i\beta(\tau_b)', \infty}(c) \\ &= \text{tr}_\alpha [(-1)^{\alpha-1} \gamma(\alpha-1, -\lambda(\tau_b))] \\ &:= \int_C \frac{\Gamma(1-\alpha)}{2\pi i} (-1)^{\alpha-1} \gamma(\alpha-1, -\lambda(\tau_b)) d\alpha \\ &= \int_{C_{+1}} \frac{\pi \csc[\pi(\alpha+1)]}{2\pi i \Gamma(\alpha+1)} (-1)^\alpha \gamma(\alpha, -\lambda(\tau_b)) d\alpha \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{(n)!} \gamma(n, -\lambda(\tau_b)) \\ &= -\sum_{n=1}^{\infty} \frac{\Gamma(n)}{\Gamma(n+1)} P(n, -\lambda(\tau_b))\end{aligned}\quad (5.2)$$

where the contour begins at  $\infty$  above the real axis, circles the point  $\{1\}$  counter-clockwise, and returns to  $\infty$  below the real axis; that is, it encloses the poles of  $\Gamma(1-\alpha)$ . The series converges absolutely since

$$\lim_{n \rightarrow \infty} \left| \frac{n!}{(n+1)!} \frac{|\gamma(n+1, -\lambda(\tau_b))|}{|\gamma(n, -\lambda(\tau_b))|} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)} \right| \lambda(\tau_b) = 0. \quad (5.3)$$

Evidently,

$$\overline{\pi_1(\tau_b+1)} - \overline{\pi_1(\tau_b)} = -\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int_{-\lambda(\tau_b)}^{-\lambda(\tau_b+1)} e^{-t} t^n dt \sim \frac{1}{\log(\tau_b)} \quad (5.4)$$

represents the density of primes at  $\tau_b$ . But we cannot infer that its inverse is the expected gap between primes since this is a conditional statement. We will return to this issue later.

Notice the significance of choosing  $\beta' = -\text{Id}'$  for the unconstrained gamma process: It forces  $\tau_b \rightarrow -\tau_b$  and it includes a phase of  $\pi$  with each event (which is associated with a pole). Roughly speaking, this calculation simply sums the positive integers appropriately constrained with a non-homogenous scaling factor and weighted by  $\Gamma(n)/\Gamma(n+1) = 1/n$ .<sup>18</sup>

A good and obvious initial choice for the scaling factor is  $\lambda(\tau_b) = \log(\tau_b)$ . Interestingly, an even better approximation is given by the somewhat curious scaling factor  $\tilde{\lambda}(\tau_b) := \log(\tilde{\tau}_b)$  where

$$\tilde{\tau}_b := \sum_{m=1}^{\infty} (-1)^{m-1} (m\tau_b)^{1/m} . \quad (5.5)$$

Numerics show that the exact prime counting function  $\pi(\tau_b)$  (apparently) randomly oscillates about  $\overline{\pi_1(\tilde{\tau}_b)}$  within a bound of  $|\overline{\pi_1(\tilde{\tau}_b)} - \pi(\tau_b)| \leq |\text{li}(\tau_b) - \pi(\tau_b)|$  for almost all  $\tau_b \in \mathbb{R}_+$  — at least up to  $\tau_b \sim O(10^{14})$ .<sup>19</sup>

Unfortunately, we are unable to give an explanation (if there is one) of why  $\log(\tilde{\tau}_b)$  is better than  $\log(\tau_b)$ . If such an explanation exists, it will likely be tied to pair correlations associated with the gamma process.

Remark that it should be possible to express incomplete gamma functions as the eigenvalues of a projection operator on the space of  $L^2(\mathbb{R}_+)$  functions with compact support with respect to a generalized Laguerre polynomial basis, and it might be profitable to think of  $\overline{\pi_1(\tau_b)}$  in that context. That is,  $\overline{\pi_1(\tilde{\tau}_b)}$  would represent the *propagator* (as a sum over point-to-point transitions represented by incomplete gamma functions) associated with a non-trivial dynamical gamma process. Perhaps the origin of  $\tilde{\tau}_b$  would be more evident from that perspective.

Similarly, the average Chebyshev theta function can be represented by

$$\begin{aligned} \overline{\theta(\tau_b)} = \text{tr}_\alpha \langle \lambda(\tau_\partial) \rangle_{\tau_b} &:= \text{tr}_\alpha \int_{T_a} \lambda(\tau_b) \mathcal{D}\gamma_{\alpha-2, -\text{Id}', \lambda(\tau_b)}(\tau) \\ &= \lambda(\tau_b) \int_{\mathcal{C}_{+2}} \frac{\Gamma(1 - (\alpha + 2))}{2\pi i} (-1)^\alpha \gamma(\alpha, -\lambda(\tau_b)) d\alpha \\ &= - \sum_{n=1}^{\infty} \frac{\lambda(\tau_b)}{(n+1)!} \gamma(n, -\lambda(\tau_b)) \\ &= - \sum_{n=1}^{\infty} \frac{\log(\tau_b)}{(n+1)!} \gamma(n, -\log(\tau_b)) . \end{aligned} \quad (5.6)$$

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<sup>18</sup>Of course, these manipulations can be bypassed (although with some loss of motivation) by simply assuming the relevant Poisson average defined in terms of the Poisson integrator that will lead directly to the summation of incomplete gamma functions in the last line.

<sup>19</sup>The numerical calculations were performed using *Mathematica* 9.0 which doesn't support prime counting beyond this order. Nevertheless, the approximation  $\pi_1(\tilde{\tau}_b)$  is rather impressive.

Consequently,

$$\begin{aligned}
\overline{\psi(\tau_b)} &:= \sum_{2^m \leq \tau_b} \overline{\theta(\tau_b^{1/m})} \\
&= - \sum_{2^m \leq \tau_b} \sum_{n=1}^{\infty} \frac{\log(\tau_b^{1/m})}{(n+1)!} \gamma(n, -\log(\tau_b^{1/m})) \\
&= - \sum_{2^m \leq \tau_b} \sum_{n=1}^{\infty} \frac{1/m \log(\tau_b)}{(n+1)!} \gamma(n, -1/m \log(\tau_b)) .
\end{aligned} \tag{5.7}$$

Like  $\overline{\pi_1(\widetilde{\tau}_b)}$ , these also provide excellent estimates for  $\tau_b \mapsto \widetilde{\tau}_b$ . Note that, since  $|\log(\widetilde{\tau}_b) \gamma(n, -\log(\widetilde{\tau}_b))| \leq |\log(\tau_b) \gamma(n, -\log(\tau_b))|$  for all  $n \in \mathbb{N}$ , then  $\overline{\theta(\widetilde{\tau}_b)} < \overline{\theta(\tau_b)}$  and  $\overline{\psi(\widetilde{\tau}_b)} < \overline{\psi(\tau_b)}$ .

This approach can be applied to twin prime counting as well. Our reasoning is heuristic. We maintain the hypothesis that the occurrence of twin prime numbers is a constrained gamma process. But now, taking pairs whose difference is  $n = 2$  will incur the twin prime constant  $C_2$  normalization according to the standard probabilistic argument. Additionally, we do not want to include the counting event between the primes so the trace should only include odd positive integers. So we should have a sum of the form

$$\overline{\pi_2(\tau_b)} \sim C_2 \sum_{n=1}^{\infty} W_2(n) \gamma(2n-1, -\lambda(\tau_b)) \tag{5.8}$$

where  $W_2(n)$  is an appropriate weight. From the prime counting case it is reasonable to guess that<sup>20</sup>  $W(n) \sim (-1)^n / (\Gamma(n+1)\Gamma(n+3))$ , and a little numerical experimentation shows that the joint probability weight is well approximated by

$$W(n) = \frac{2(-1)^n}{\Gamma(n+1)\Gamma(n+3)} . \tag{5.9}$$

Continuing this heuristic for arbitrary prime doubles  $\overline{\pi_{2i}(\tau_b)}$  within an interval  $2i \leq \tau_b - 2$ , the normalizing constant becomes

$$C_{2i} = C_2 \prod_{p|i} \frac{p-1}{p-2} \tag{5.10}$$

for prime numbers  $p > 2$ . Inside the sum, the key is to group all integers between the associated primes together. That is, we still sum over odd positive integers to exclude the *collection* of numbers between the two primes, and the weight factor for

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<sup>20</sup>The phase factor  $(-1)^n$  shows up because the phases of the two primes give a joint contribution of  $(-1)^{2n}$  which no longer cancels the  $(-1)^n$  coming from the residue at the poles.

the joint probability stays the same. In general then, the prime double hypothesis is

$$\begin{aligned}\overline{\pi_{2i}(\tau_b)} &= C_{2i} \sum_{n=1}^{\infty} \frac{2(-1)^n}{\Gamma(n+1)\Gamma(n+3)} \gamma(2n-1, -\lambda(\tau_b)) \\ &= C_{2i} \sum_{n=1}^{\infty} \frac{2 \cos[\pi(\frac{n+1}{2} + 2)]}{\Gamma(\frac{n+1}{2} + 1)\Gamma(\frac{n+1}{2} + 3)} \gamma(n, -\lambda(\tau_b)) , \quad \tau_b - 2 > 2i \in \mathbb{N} .\end{aligned}\tag{5.11}$$

Note that only the normalizing constant depends on  $i$ , and the series converges absolutely for finite  $\tau_b$ ;

$$\lim_{n \rightarrow \infty} \left| \frac{\Gamma(n+1)\Gamma(n+3)}{\Gamma(n+2)\Gamma(n+4)} \right| \left| \frac{|\gamma(2n, -\lambda(\tau_b))|}{|\gamma(2n-1, -\lambda(\tau_b))|} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)(n+3)} \right| \lambda(\tau_b) = 0 .\tag{5.12}$$

However, since  $\lim_{\tau_b \rightarrow \infty} \gamma(2n-1, -\lambda(\tau_b)) = -\Gamma(2n-1)$ , and the series

$$\{a_n\} := \frac{2(-1)^n \Gamma(2n-1)}{\Gamma(n+1)\Gamma(n+3)}\tag{5.13}$$

does not converge to zero, then  $\overline{\pi_{2i}(\tau_b)}$  diverges as  $\tau_b \rightarrow \infty$ . Moreover, although  $W(n)$  was essentially guessed with some numerical justification, the exact weight  $\widehat{W}(n)$  (if it is different)<sup>21</sup> is certainly bounded by

$$\frac{(-1)^n}{\Gamma(n+3)\Gamma(n+3)} < \widehat{W}(n) < \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1)}\tag{5.14}$$

and the sums associated with both bounds diverge as  $\tau_b \rightarrow \infty$ . Therefore, given the hypothesis of the non-homogenous gamma process for locating joint prime numbers, we conclude that

$$\lim_{\tau_b \rightarrow \infty} \overline{\pi_{2i}(\tau_b)} \rightarrow \infty \quad \forall i \in \mathbb{N}\tag{5.15}$$

for any  $W(n)$  between the bounds — including  $\widehat{W}(n)$ .

Owing to its probabilistic foundation, the prime double hypothesis cannot be confirmed unconditionally. However, given the success of the average prime counting function  $\overline{\pi_1(\tau_b)}$ , it appears likely that the hypothesis is correct. In particular, if we accept it for at least  $i = 1$ , then verification of the twin prime counting conjecture follows immediately — albeit conditionally — since the sum diverges with  $\tau_b$ .

Of course, one might argue that the hypothesis is just an alternative to the Hardy-Littlewood twin prime conjecture. However, the hypothesis is not asymptotic. With it we can statistically verify the Goldbach conjecture<sup>22</sup>:

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<sup>21</sup>The agreement between the expected and actual prime double counting numbers is good but not as impressive as for single primes. We conjecture that  $W(n)$  is indeed the correct weight and the degradation comes from  $\tilde{\tau}_b$  due to pair-pair correlations.

<sup>22</sup>I acknowledge Jean-Pierre Zablitz for bringing the Goldbach conjecture to my attention.

**Theorem 5.1** *If the occurrence of prime doubles is a non-homogenous gamma process, then every even number greater than 2 is asymptotically almost surely the sum of two primes.*

*Sketch of Proof:* Assume the contrary. Then there exists a  $2\tau_o$  that is not the sum of two primes. Clearly,  $\tau_o$  cannot be prime. Further,  $\tau_o$  cannot be ‘straddled’ by a prime double  $(p, p+2i)$  with  $p+i = \tau_o$  for some  $i \in \{1, \dots, \tau_o - 1\}$  since otherwise  $p + [p+2i] = (\tau_o - i) + [(\tau_o - i) + 2i] = 2\tau_o$ .

But according to the constrained gamma process hypothesis, the probability density of prime doubles straddling the point  $\tau_o$  is given by the absolutely converging series

$$P_i(\tilde{\tau}_o) = C_{2i} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+3)} \frac{\Delta_+(n, \tilde{\tau}_o) + \Delta_-(n, \tilde{\tau}_o)}{2i} \quad (5.16)$$

where

$$\Delta_{\pm}(n, \tau_o) := [\pm\gamma(2n-1, -\lambda(\tau_o \pm 1)) \mp \gamma(2n-1, -\lambda(\tau_o))] . \quad (5.17)$$

So the expected number of prime doubles that straddle  $\tau_o$  is given by

$$S(\tilde{\tau}_o) := \sum_{i=1}^{\tau_o-1} \frac{\pi_{2i}(2\tilde{\tau}_o)}{\pi_{2i}(2\tilde{\tau}_o)} P_i(\tilde{\tau}_o) . \quad (5.18)$$

Now, for sufficiently large  $\tau_o$  the expected number goes like  $S(\tau_o) \sim \tau_o/(\log^4(\tau_o))$ . Moreover, since  $\tau_o/(\log^4(\tau_o))$  is monotonically increasing for sufficiently large  $\tau_o$ , it only takes calculating  $S(\tau_o)$  for a few small  $\tau_o$  to see that  $S(\tau_o) > 1$  for all  $\tau_o \geq 6$ . Since the probability that  $\tau_o$  is straddled by at least one prime double is  $1 - e^{-S(\tau_o)}$ , we have a contradiction asymptotically almost surely.  $\square$

One can check explicitly up to some sufficiently large  $\tau_b = 2\tau_o$  that the probability of a contradiction is essentially almost sure. For example at  $\tau_b = 10^9$  we find the expected number of straddling prime doubles  $S(\tilde{\tau}_b) > 29000$  where we used the *underestimate*  $\sum_i C_{2i}/i \approx 1$  for simplicity. So the probability that the next even integer is not the sum of two primes is less than about  $10^{-12500}$ . It is perhaps disconcerting that the conjecture cannot be settled with certainty by this argument, but it is comforting that the probability that it is false — beyond where one is willing to explicitly check — decreases exponentially like  $e^{-c\tau_b/(\log(\tau_b))^4}$  with  $c \sim O(1)$  a positive constant.<sup>23</sup>

One final implication: Since the probability associated with prime doubles only depends on the gap between them through  $C_{2i}$ , the probability of twin primes at an interval  $[\tau_b - 1, \tau_b + 1]/2$  is the joint distribution to use for the conditional probability of two primes being separated by a gap. So the expected gap between prime  $p_1$  and  $p_2$  given  $p_1$  is  $P(p_1 + 1)^{-1}$ , and it is easy to establish that  $P(p)^{-1} \sim \log(p)^2$ . Hence Cramér’s conjecture is true on average.

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<sup>23</sup>If the prime numbers are randomly distributed, then it is plausible that a proof of the Goldbach conjecture with certainty is not possible. This would help explain its resistance to attacks.



## 5.2 Bound estimates

Note that for  $\tau_b > 0$ ,

$$\frac{d\tilde{\tau}_b}{d\tau_b} = 1 - \frac{1}{\sqrt{2\tau_b}} + \sum_{m=3}^{\infty} (-1)^{m-1} (m\tau_b)^{1/m-1} > 1 - \frac{1}{\sqrt{2\tau_b}}. \quad (5.19)$$

On the other hand,

$$\begin{aligned} \tilde{\tau}_b &= \tau_b + \sum_{m=2}^{\infty} (-1)^{m-1} (m\tau_b)^{1/m} < \tau_b + \sqrt{2\tau_b} \sum_{m=2}^{\infty} (-1)^{m-1} \\ &= \tau_b - \sqrt{2\tau_b} \sum_{m=1}^{\infty} (-1)^{m-1} \\ &= \tau_b - \frac{\sqrt{2\tau_b}}{2} < \tau_b + \sqrt{2\tau_b} \end{aligned} \quad (5.20)$$

where Cesàro summation was used. Hence,  $\tilde{\tau}_b \approx \tau_b - 3\sqrt{2\tau_b}/4$  and  $|\tilde{\tau}_b - \tau_b| < \sqrt{2\tau_b}$  for  $\tau_b \geq 2$ .

For the Chebyshev functions with  $\tau_b \geq 2$ ,

$$\begin{aligned} 0 < \frac{d\overline{\theta(\tau_b)}}{d\tau_b} &= 1 - \frac{1}{\log(\tau_b)} \left(1 - \frac{1}{\tau_b}\right) - \sum_{n=1}^{\infty} \frac{1/\tau_b}{(n+1)!} \gamma(n, -\log(\tau_b)) \\ &< 1 - \frac{1}{\log(\tau_b)} \left(1 - \frac{1}{\tau_b}\right) \\ &< 1, \end{aligned} \quad (5.21)$$

and

$$\begin{aligned} 0 < \frac{d\overline{\psi(\tau_b)}}{d\tau_b} &= \sum_{2^m \leq \tau_b} \overline{\theta(\tau_b^{1/m})} \frac{\tau_b^{1/m-1}}{m} \\ &= \sum_{2^m \leq \tau_b} \frac{\tau_b^{1/m-1}}{m} \left(1 - \frac{1}{\log(\tau_b^{1/m})} \left(1 - \frac{1}{\tau_b^{1/m} \log(\tau_b^{1/m})}\right)\right) \\ &< \left(1 - \frac{1}{\log(\tau_b)}\right) \sum_{2^m \leq \tau_b} \frac{\tau_b^{1/m-1}}{m}, \quad \tau_b \geq 9 \\ &= \left(1 - \frac{1}{\log(\tau_b)}\right) \frac{1}{\tau_b} \sum_{2^m \leq \tau_b} \frac{\tau_b^{1/m}}{m} \\ &< \left(1 - \frac{1}{\log(\tau_b)}\right) \left(1 + \frac{1}{\log(\tau_b)}\right) \\ &< 1. \end{aligned} \quad (5.22)$$

For the prime counting function with  $\tau_b \geq 2$ ,

$$\frac{d\overline{\pi_1(\tau_b)}}{d\tau_b} = \frac{\tau_b - 1}{\tau_b \log(\tau_b)} \quad (5.23)$$

yielding

$$\overline{\pi_1(\tau_b)} - \overline{\pi_1(2)} = \text{li}(\tau_b) - \log(\log(\tau_b)) + \log(\log(2)) - \text{li}(2) . \quad (5.24)$$

Consequently  $\overline{\pi_1(\tau_b)} < \text{li}(\tau_b)$  and  $\left| \text{li}(\tau_b) - \overline{\pi_1(\tau_b)} \right| < \log(\log(\tau_b)) < \log^{1/2}(\tau_b)$ .

So for  $\tau_b$  sufficiently large, we have the estimates

$$\left| \tilde{\theta}(\tau_b) - \tau_b \right| < \sqrt{2\tau_b} \quad (5.25)$$

$$\left| \tilde{\psi}(\tau_b) - \tau_b \right| < \sqrt{2\tau_b} \quad (5.26)$$

$$\left| \tilde{\text{li}}(\tau_b) - \tilde{\pi}_1(\tau_b) \right| < \sqrt{\log(\tilde{\tau}_b)} \quad , \quad \left| \text{li}(\tau_b) - \pi_1(\tau_b) \right| < \sqrt{\log(\tau_b)} \quad (5.27)$$

using the notation  $\tilde{f}(\tau_b) := \overline{f(\tilde{\tau}_b)}$ . Loosening the bound to  $\sqrt{4\tau_b}$  renders the estimate for  $\psi(\tau_b)$  valid for all  $\tau_b \geq 2$ .

These estimates enable comparison between the average and exact prime counting function:

**Proposition 5.1** *Assume that prime counting is a non-homogenous gamma process. For  $\tau_b \geq 8$ , let*

$$1 < N \leq \log(\tau_b) \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2\tau_b}} \right) \quad (5.28)$$

and

$$0 < M \leq \sqrt{\log(\tau_b)} \left( \frac{1}{\sqrt{2}} - \frac{1}{\log(\tau_b)} \right) . \quad (5.29)$$

Then

$$\left| \pi(\tau_b) - \overline{\pi_1(\tau_b)} \right| < \left| \pi(\tau_b) - \text{li}(\tau_b) \right| < \sqrt{\tau_b \log(\tau_b)} \quad (5.30)$$

and

$$\left| \pi(\tau_b) - \tilde{\pi}_1(\tau_b) \right| < \sqrt{\tau_b} \quad (5.31)$$

asymptotically almost surely.<sup>24</sup>

*Proof:* The variance for the trivial gamma process is equal to its mean. So the variance of the constrained process<sup>25</sup> goes like

$$\sigma^2 \sim \overline{\pi_1(\tau_b)} < \text{li}(\tau_b) < \frac{2\tau_b}{\log(\tau_b)} , \quad \forall \tau_b \geq 8 . \quad (5.32)$$

Then, for a confidence interval depending on  $N$ ,

$$\left| \pi(\tau_b) - \overline{\pi_1(\tau_b)} \right| \leq N\sigma < N\sqrt{\frac{2\tau_b}{\log(\tau_b)}} . \quad (5.33)$$

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<sup>24</sup>Numerics appear to indicate even tighter bounds  $\left| \pi(\tau_b) - \overline{\pi_1(\tau_b)} \right| < N\sqrt{\tau_b/\log(\tau_b)}$  and  $\left| \pi(\tau_b) - \tilde{\pi}(\tau_b) \right| < M\sqrt{\tau_b/\log(\tau_b)}$  for constant  $N$  and  $M$ .

<sup>25</sup>This is essentially the variance of the number of events for the corresponding Poisson process.

Hence

$$|\pi(\tau_b) - \text{li}(\tau_b)| \leq \left| \pi(\tau_b) - \overline{\pi_1(\tau_b)} \right| + \left| \overline{\pi_1(\tau_b)} - \text{li}(\tau_b) \right| < N \sqrt{\frac{2\tau_b}{\log(\tau_b)}} + \sqrt{\log(\tau_b)}. \quad (5.34)$$

Similarly,

$$|\pi(\tau_b) - \tilde{\pi}(\tau_b)| \leq \left| \pi(\tau_b) - \overline{\pi_1(\tau_b)} \right| + \left| \overline{\pi_1(\tau_b)} - \tilde{\pi}(\tau_b) \right| < M \sqrt{\frac{2\tau_b}{\log(\tau_b)}} + \frac{\sqrt{2\tau_b}}{\log(\tau_b)}. \quad (5.35)$$

Since the upper bounds on  $N$  and  $M$  increase monotonically to  $\infty$ , the confidence levels for the estimates increase asymptotically to 100%.  $\square$

This proposition has obvious implications for the distribution of primes in general and the Riemann hypothesis in particular. But much like the case of the Golbach conjecture, it only statistically confirms the hypothesis asymptotically almost surely. Again, if the prime numbers are randomly distributed, then perhaps one cannot hope to do better than this by relating zeroes of Riemann zeta to prime counting functions. However, Riemann zeta as a function need not be tied to prime counting, and it is reasonable to expect the Riemann hypothesis can be verified unconditionally and with certainty by other means.

## 6 Conclusions

Constrained physical systems were studied from a function space perspective using newly developed functional integration tools. The tools rely on the notions of conditional and conjugate integrators — the analogs of conditional and conjugate probability distributions in Bayesian inference theory. These notions show the well-known Gaussian functional integrals to be only part of the picture: To describe constrained systems, one must be able to manipulate functional integrals over constrained function spaces with conjugate integrator families.

Using the constrained functional integral concept, well-known QM results were re-derived efficiently at the functional level. Additionally, the framework allowed construction of a model for various counting functions associated with prime numbers that give improved numerical estimates and, hopefully, a basis for understanding prime distributions. The examples analyzed here point to the utility of gamma and Poisson integrator families, but it is likely that other probability distribution analogs will be useful.

No attempt was made to develop methods to calculate non-trivial gamma functional integrals. That  $Z(\tau')$  is comprised of the incomplete gamma function and functional determinants *and* it is defined for complex parameters, points to broad complexity. Evidently the study of  $\mathcal{D}\gamma$  is an involved but important project. The perturbation expansion notwithstanding, the gamma functional integral can be expected to yield new calculation techniques.

It would be fruitful to extend the concepts developed in this article beyond simple QM. In particular, the domain of  $x$  and  $\tau$  can be altered in obvious ways to include quantum fields and loops. Similarly, the domain of  $X_a$  can be extended to include matrix-valued functions — opening the door to matrix QM. Together with the complex Gaussian integrator and the complex nature of the gamma integrator, such extensions would appear to offer broad applicability and significant potential.

## A CDM scheme

The CDM scheme ([1], [2], [3]) defines functional integrals in terms of the data  $(B, \Theta, Z, \mathcal{F}(B))$ .

Here  $B$  is a separable (usually) infinite dimensional Banach space with a norm  $\|b\|$  where  $b \in B$  is an  $L^{2,1}$  map  $b : [t_a, t_b] \in \mathbb{R} \rightarrow \mathbb{M}$  with  $\mathbb{M}$  an  $m$ -dimensional paracompact differentiable manifold. The dual Banach space  $B' \ni b'$  is a space of linear forms such that  $\langle b', b \rangle_B \in \mathbb{C}$  with an induced norm given by

$$\|b'\| = \sup_{b \neq 0} |\langle b', b \rangle| / \|b\| .$$

Assume  $B'$  is separable. Then  $B'$  is Polish and consequently admits complex Borel measures  $\mu$ .

$\Theta$  and  $Z$  are bounded,  $\mu$ -integrable functionals  $\Theta : B \times B' \rightarrow \mathbb{C}$  and  $Z : B' \rightarrow \mathbb{C}$ . The functional  $\Theta(b, \cdot)$  can be thought of as the functional analog of a probability distribution function and  $Z(b')$  the associated characteristic functional.

The final datum is the space of integrable functionals  $\mathcal{F}(B)$  consisting of functionals  $F(b)$  defined relative to  $\mu$  by

$$F_\mu(b) := \int_{B'} \Theta(b, b') d\mu(b') . \quad (\text{A.1})$$

If  $\mu \mapsto F_\mu$  is injective, then  $\mathcal{F}(B)$  is a Banach space endowed with a norm  $\|F_\mu\|$  defined to be the total variation of  $\mu$ .

These data are used to define an integrator  $\mathcal{D}_{\Theta, Z}b$  on  $B$  by

$$\int_B \Theta(b, b') \mathcal{D}_{\Theta, Z}b := Z(b'), \quad (\text{A.2})$$

This defines an integral operator  $\int_B$  on the normed Banach space  $\mathcal{F}(B)$ ;

$$\int_B F_\mu(b) \mathcal{D}_{\Theta, Z}b := \int_{B'} Z(b') d\mu(b') . \quad (\text{A.3})$$

And the integral operator  $\int_B$  is a bounded linear form on  $\mathcal{F}(B)$  with

$$\left| \int_B F_\mu(b) \mathcal{D}_{\Theta, Z} b \right| \leq \|F_\mu\|. \quad (\text{A.4})$$

## B Integrator families

### B.0.1 Gaussian family

**Definition B.1** Let  $X_a$  be the space of  $L^{2,1}$  pointed functions  $x : [t_a, t_b] \subseteq \mathbb{R} \rightarrow \mathbb{X}$  such that  $x(t_a) =: x_a \in \mathbb{X}$  with  $\mathbb{X}$  real. A Gaussian family of integrators  $\mathcal{D}\omega_{\bar{x}, Q}(x)$  is characterized by<sup>26</sup>

$$\begin{aligned} \Theta(x, x') &= \exp\{2\pi i \langle x', x \rangle\} \\ Z_{\bar{x}, W}(x') &= \sqrt{s} \text{Det}[W^{1/2}] e^{2\pi i \langle x', \bar{x} \rangle - \pi s W(x')} \end{aligned} \quad (\text{B.1})$$

where  $\langle x', x \rangle \in \mathbb{R}$ ,  $s \in \{1, i\}$ , the mean path<sup>27</sup>  $\bar{x}$  has boundary conditions  $\bar{x}(t_a) = x_a$  and  $\dot{\bar{x}}(t_b) = \dot{x}_b$ , the functional determinant is assumed to be well-defined, and the variance

$$W(x'_1, x'_2) = \frac{1}{2} \{ \langle x'_1, Gx'_2 \rangle + \langle x'_2, Gx'_1 \rangle \} =: \langle x', Gx' \rangle_{\{1,2\}} \quad (\text{B.2})$$

where the covariance matrix  $G : X'_a \rightarrow X_a$  with  $\Re(sG)$  is non-negative definite. Associated with the variance is a precision

$$Q(x_1, x_2) = \langle Dx, x \rangle_{\{1,2\}} - B(x_1, x_2) \quad (\text{B.3})$$

with a boundary form

$$B(x_1, x_2) := \langle Dx_1, x_2 \rangle - \langle D^\dagger x_2, x_1 \rangle \quad (\text{B.4})$$

where  $D = G^{-1}$  and  $D^\dagger$  is its time-reversed adjoint<sup>28</sup> relative to a Hilbert structure on  $X_a$ .

The integrator family is defined in terms of the primitive Gaussian integrator  $\mathcal{D}x$ ;

$$\mathcal{D}\omega_{\bar{x}, Q}(x) := e^{-(\pi/s)Q(x-\bar{x})} \mathcal{D}x \quad (\text{B.5})$$

---

<sup>26</sup>This definition uses a different normalization from the usual Gaussian integrator in the CDM scheme. Both definitions are valid: we choose this normalization because it seems more consistent with definitions of other integrator families and it highlights the role of the functional determinant.

<sup>27</sup>To remind; the mean path is also a critical path in this case since  $Q$  is quadratic. Also, most physics applications of functional integrals take place in unbounded configuration space where it is customary to put  $\bar{x} = 0$ . However, this seemingly innocuous simplification cannot be taken for conditional functional integrals since  $\bar{x}$  plays an essential role as we will see in §4. This could be anticipated from the analogy with sufficient statistics.

<sup>28</sup>The time parameter runs ‘backwards’ for  $D^\dagger$  because the order  $1 \rightarrow 2$  has been reversed. For example, if  $D = d^2/dt^2$ , then  $B(x_1, x_2) = 1/2 \int \dot{x}_1 \dot{x}_2 dt = 0$  when  $x(t_a) = \dot{x}(t_b) = 0$ . Otherwise  $B(x_1, x_2) = 1/2(x_1 \dot{x}_2|_{t_a}^{t_b} - \dot{x}_1 x_2|_{t_a}^{t_b})$  so  $B(x) = x \dot{x}|_{t_a}^{t_b} \neq 0$ .

where  $\mathcal{D}x$  is characterized by

$$\Theta(x, x') = \exp\{2\pi i \langle x', x \rangle - (\pi/s) \text{Id}(x)\} \ ; \ Z(x') = \sqrt{s} e^{-\pi s \text{Id}(x')} . \quad (\text{B.6})$$

The primitive integrator can be thought of as a Gaussian with zero mean and trivial covariance, i.e.  $\mathcal{D}\omega_{0, \text{Id}}(x)$ . Loosely,  $\mathcal{D}x$  is the integrator analog of the Lebesgue measure on  $\mathbb{R}^n$ . Note that  $W$  (and hence  $\text{Det } W$ ), inherits the boundary conditions imposed on  $\bar{x}$ , and note the normalizations

$$\int_{X_a} e^{-(\pi/s) \text{Id}(x)} \mathcal{D}x = \sqrt{s} \quad (\text{B.7})$$

and

$$\int_{X_a} \mathcal{D}\omega_{\bar{x}, Q}(x) = \sqrt{s} \text{Det}[W^{1/2}] . \quad (\text{B.8})$$

The symmetry between the functional form of  $Z(x')$  and the integrand motivates the standard practice in quantum field theory of defining the effective action functional  $\Gamma : X_a \times X'_a \rightarrow \mathbb{C}$  by

$$-(\pi/s) \Gamma(\bar{x}, x') := 2\pi i \langle x', \bar{x} \rangle - \pi s W(x') . \quad (\text{B.9})$$

Clearly the exponentiated effective action is nothing other than the characteristic functional of the Gaussian integrator. Notice that, if  $D$  is self-adjoint,  $\Gamma(\bar{x}, x') = \Gamma(\bar{x}, Dx) = Q(x, \bar{x})$ . In particular,  $\Gamma(\bar{x}) = Q(\bar{x})$  where  $\bar{x}' := D\bar{x}$ , and the functional integral is easily evaluated once  $\bar{x}$  is known.

The desire to handle action functionals more general than quadratic functionals suggests defining more general exponential-type integrators in the usual way;

**Definition B.2** Let  $X_a$  be the space of  $L^{2,1}$  pointed functions  $x : [t_a, t_b] \subseteq \mathbb{R} \rightarrow \mathbb{X}$  such that  $x(t_a) =: x_a \in \mathbb{X}$  with  $\mathbb{X}$  real. An exponential family of integrators  $\mathcal{D}\omega_{\bar{x}, S}(x)$  is characterized by

$$\begin{aligned} \Theta(x, x') &= \exp\{2\pi i \langle x', x \rangle\} \\ Z_{\bar{x}, W_S}(x') &= \sqrt{s} \text{Det}[W_S^{1/2}] e^{2\pi i \langle x', \bar{x} \rangle - \pi s W_S(x')} \end{aligned} \quad (\text{B.10})$$

where  $\langle x', x \rangle \in \mathbb{R}$ ,  $s \in \{1, i\}$ , the mean path  $\bar{x}$  has boundary conditions  $\bar{x}(t_a) = x_a$  and  $\dot{\bar{x}}(t_b) = \dot{x}_b$ , and  $W_S(x') = \langle x', G_S x' \rangle$  where  $G_S$  is the connected Green's function. The integrator family is defined in terms of the primitive Gaussian integrator  $\mathcal{D}x$ ;

$$\mathcal{D}\omega_{\bar{x}, S}(x) := e^{-(\pi/s) S(x - \bar{x})} \mathcal{D}x . \quad (\text{B.11})$$

The effective action<sup>29</sup> associated with this integrator family is

$$-(\pi/s) \Gamma_S(\bar{x}, x') := 2\pi i \langle x', \bar{x} \rangle - \pi s W_S(x') , \quad (\text{B.12})$$

---

<sup>29</sup>Remark that, evaluating at  $\bar{x}' := D\bar{x}$  for self-adjoint  $D$ , we have  $\Gamma_S(\bar{x}) = S(\bar{x})$ . That is, the effective action has the same functional form as the action *relative to*  $\mathcal{D}\omega_{\bar{x}, S}(x)$ . Consequently, the functional integral is readily evaluated *if  $\bar{x}$  is known*. But  $\bar{x}$  is hard to find for  $S$ . In quantum field theory, functional integrals are defined relative to  $\mathcal{D}\omega_{\bar{x}, Q}(x)$ ; in which case  $\bar{x}$  is easy to find but then  $\Gamma$  becomes a functional power series in  $S$ . In fact, the whole effective action technique is a realization of this trade-off — finding the moments of an unknown probability distribution associated with  $(\bar{x}_S, S)$  relative to a Gaussian distribution associated with  $(\bar{x}_Q, Q)$ .

and the inverse of  $G_S$  is defined by

$$\int D_S(t, t') G_S(t', s) dt' := \int \frac{\delta^2 \Gamma_S(x)}{\delta x(t) \delta x(t')} \frac{\delta^2 W_S(x')}{\delta x'(t') \delta x'(s)} dt' = \delta(t, s) . \quad (\text{B.13})$$

To see how conditional Gaussian integrators work, form the product space  $X_a \times Y_a$ . Suppose a Gaussian integrator on  $X_a \times Y_a$  is characterized by a positive definite quadratic form  $\tilde{Q}$  with mean  $\tilde{m}$  and vanishing boundary term. Put  $\tilde{m} = (\bar{x}, \bar{y})$  and

$$\tilde{G} = \begin{pmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{pmatrix} . \quad (\text{B.14})$$

Then<sup>30</sup>

$$\tilde{Q}((x, y) - \tilde{m}) = Q_X(x - m_{x|y}) + Q_Y(y - \bar{y}) \quad (\text{B.15})$$

where  $Q_Y(y_1, y_2) = \langle D_{yy} y_1, y_2 \rangle$ ,

$$Q_X(x_1, x_2) = \langle (G_{xx} - G_{xy} D_{yy} G_{yx})^{-1} x_1, x_2 \rangle \quad (\text{B.16})$$

and

$$m_{x|y} = \bar{x} + G_{xy} D_{yy} (y - \bar{y}) . \quad (\text{B.17})$$

So the Gaussian integrator on  $X_a \times Y_a$  is

$$\mathcal{D}\omega_{\tilde{m}, \tilde{Q}}(x, y) := e^{-(\pi/s)\tilde{Q}((x, y) - \tilde{m})} \mathcal{D}(x, y) . \quad (\text{B.18})$$

On the other hand,

$$\mathcal{D}\omega_{\bar{y}, Q_Y}(y) := e^{-(\pi/s)Q_Y(y - \bar{y})} \mathcal{D}y . \quad (\text{B.19})$$

Therefore, the *conditional Gaussian integrator* is

$$\mathcal{D}\omega_{m_{x|y}, Q_{X|Y}}(x|y) := e^{-(\pi/s)Q_X(x - m_{x|y})} \mathcal{D}(x, y) \quad (\text{B.20})$$

which yields

$$\int_{B_a} \mathcal{D}\omega_{m_{x|y}, Q_{X|Y}}(x|y) = e^{-(\pi/s)\tilde{Q}_{\partial}(m_{x|y})} \text{Det} \left[ \left( \frac{Q_X + Q_Y}{Q_Y} \right)^{-1/2} \right] \quad (\text{B.21})$$

with  $\tilde{Q}_{\partial}(m_{x|y}) := \tilde{\Gamma}(m_{x|y}, Dm_{x|y})$ .

In particular, let  $M : X_a \rightarrow Y_a$  be a homeomorphism such that  $Q_1 = Q_2 \circ M$ . If  $Y_a = X_a$  then  $G_{xy} = G_{yx} = 0$  since the  $x$  are independent Gaussian variables. Also  $\bar{y} = M(\bar{x})$ . Then formally,

$$\mathcal{D}\omega_{\bar{x}, Q_1}(x) = \frac{\mathcal{D}\omega_{\bar{x}, Q_1}(x|y)}{\mathcal{D}\omega_{\bar{y}, Q_2}(x|y)} \mathcal{D}\omega_{\bar{y}, Q_2}(y) . \quad (\text{B.22})$$

---

<sup>30</sup>It can be shown that  $Q_X$  and  $Q_Y$  are positive definite since  $\tilde{G}$  is positive definite.

But<sup>31</sup>

$$\frac{\mathcal{D}\omega_{\bar{x},Q_1}(x|y)}{\mathcal{D}\omega_{\bar{y},Q_2}(x|y)} \sim \frac{e^{(\pi/s)Q_1(\bar{x})}}{e^{(\pi/s)Q_2(\bar{y})}} \text{Det} \left[ \left( \frac{Q_2}{Q_1} \right)^{-1/2} \right] \quad (\text{B.23})$$

so we get the standard result for a change of covariance;

$$\int_{X_a} e^{-(\pi/s)Q_2(x-M(\bar{x}))} \mathcal{D}_1 x = \text{Det} \left[ \left( \frac{Q_2}{Q_1} \right)^{-1/2} \right] \quad (\text{B.24})$$

where  $\mathcal{D}_1 x$  is the primitive integrator on  $X_a$ . Obviously the same condition holds for  $1 \leftrightarrow 2$  with  $\mathcal{D}_2 x$  the primitive integrator on  $M(X_a)$ .

### B.0.2 Complex Gaussian family

The previous subsection took the parameter  $s \in \{1, i\}$ .<sup>32</sup> This restriction can be lifted by defining a complex Gaussian integrator.

**Definition B.3** Let  $Z_a^2$  be the space of  $L^{2,1}$  pointed functions  $(z, \underline{z}) : [t_a, t_b] \subseteq \mathbb{R} \rightarrow \mathbb{M}^{\mathbb{C}}$  such that  $(z, \underline{z})(t_a) =: (z_a, \underline{z}_a) \in \mathbb{M}^{\mathbb{C}}$  with  $\mathbb{M}^{\mathbb{C}}$  a complex manifold. A complex Gaussian family of integrators  $\mathcal{D}\omega_{\bar{w},Q^{\mathbb{C}}}(w)$  on  $W_a \equiv Z_a^2$  is characterized by

$$\begin{aligned} \Theta(w, w') &= \exp\{2\pi i \langle w', w \rangle\} \\ Z_{\bar{w},W^{\mathbb{C}}}(w') &= \text{Det}[W^{\mathbb{C}1/2}] e^{2\pi i \langle w', \bar{w} \rangle - \pi W^{\mathbb{C}}(w')} \end{aligned} \quad (\text{B.25})$$

where  $w := (z, \underline{z}) \in W_a$ ,  $w' = (z', \underline{z}') \in W'_a$ , and  $\langle w', w \rangle \in \mathbb{C}$ . The complexified variance  $W^{\mathbb{C}}(w'_1, w'_2) = \langle w'_1, G^{\mathbb{C}} w'_2 \rangle$  where the complex covariance matrix  $G^{\mathbb{C}}$  has the block form

$$G^{\mathbb{C}} = \begin{pmatrix} G_{\underline{z}z} & G_{\underline{z}\underline{z}} \\ G_{zz} & G_{z\underline{z}} \end{pmatrix} \quad (\text{B.26})$$

with  $\Re(\langle w'_1, G^{\mathbb{C}} w'_2 \rangle) \geq 0$  and  $G^{\mathbb{C}}$  not necessarily Hermitian.<sup>33</sup> As in the real case, put

$$\mathcal{D}\omega_{\bar{w},Q^{\mathbb{C}}}(w) = e^{-\pi Q^{\mathbb{C}}(w-\bar{w})} \mathcal{D}w \quad (\text{B.27})$$

where  $\mathcal{D}w$  is characterized by

$$\Theta(w, w') = \exp\{2\pi i \langle w', w \rangle - \pi \text{Id}(w)\} \ ; \ Z(w') = e^{-\pi \text{Id}(w')} \ . \quad (\text{B.28})$$

At the level of functional integrals, evidently there is little difference between the real and complex Gaussian families. The value in the complex case comes when the domain of integration is localized yielding complex line integrals. The complex Gaussian can be extended to the more general complex exponential for  $S^{\mathbb{C}}$  exactly as in the real case.

<sup>31</sup>The ratio of the phase factors is obviously trivial since  $\bar{y} = M(\bar{x})$ .

<sup>32</sup>That Gaussian integrators based on non-negative definite *real*  $G$  can be defined for  $s \in \{1, i\}$  reflects the validity of the Schrödinger $\leftrightarrow$ diffusion correspondence through analytic continuation. However, analytic continuation does not maintain this correspondence in general. It is natural to conjecture that the analytic continuation Schrödinger $\leftrightarrow$ diffusion correspondence will break down precisely when  $G_{zz}$  and/or  $G_{\underline{z}\underline{z}}$ , defined below, do not vanish.

<sup>33</sup>If  $\underline{z} = z^*$  then  $(G^{\mathbb{C}})^{\dagger} = G^{\mathbb{C}}$ .



### B.0.3 Gamma family

**Definition B.4** Let  $T_a$  be the space of  $L^{2,1}$  pointed functions  $\tau : [t_a, t_b] \rightarrow \mathcal{C} \subseteq \mathbb{C}$  such that  $\tau(t_a) = 0$ . Its dual space  $T'_a$  is the space of linear forms  $\mathcal{L}(T_a, \mathbb{C})$ . A lower gamma family of integrators  $\mathcal{D}\gamma_{\alpha, \beta', \tau_\partial}(\tau)$  is characterized by<sup>34</sup>

$$\Theta(\tau, \tau') = \exp\{i\langle \tau', \tau \rangle\} \ ; \ Z_{\alpha, \beta', \tau_\partial}(\tau') = \frac{\gamma(\alpha, \overline{\tau_\partial})}{\text{Det}(\beta' - i\tau')^\alpha} \quad (\text{B.29})$$

where  $\alpha \in \mathbb{C}$ ,  $\beta' \in T'_a$ ,  $\overline{\tau_\partial} := \langle \beta', \tau_\partial \rangle$  for some fixed  $\tau_\partial \in T_a$ , the functional determinant is assumed well-defined, and  $\gamma(\alpha, \overline{\tau_\partial})$  is the lower incomplete gamma function given by

$$\gamma(\alpha, \overline{\tau_\partial}) = \Gamma(\alpha) e^{-\overline{\tau_\partial}} \sum_{n=0}^{\infty} \frac{(\overline{\tau_\partial})^{\alpha+n}}{\Gamma(\alpha+n+1)} . \quad (\text{B.30})$$

The integrator family is defined in terms of the primitive gamma integrator by

$$\mathcal{D}\gamma_{\alpha, \beta', \tau_\partial}(\tau) := \tau^\alpha e^{-\langle \beta', \tau \rangle} \mathcal{D}\tau \quad (\text{B.31})$$

where  $\tau^\alpha$  is defined point-wise by  $\tau^\alpha(t) = \tau(t)^\alpha$  and  $\mathcal{D}\tau$  is characterized by

$$\Theta(\tau, \tau') = \exp\{i\langle \tau', \tau \rangle - \langle \text{Id}', \tau \rangle\} \ ; \ Z(\tau') = 1 . \quad (\text{B.32})$$

An upper gamma family of integrators  $\mathcal{D}\Gamma_{\alpha, \beta', \tau_\partial}(\tau)$  is defined similarly where

$$\Gamma(\alpha, \overline{\tau_\partial}) = \Gamma(\alpha) - \gamma(\alpha, \overline{\tau_\partial}) \quad (\text{B.33})$$

is the upper incomplete gamma function.

The primitive gamma integrator  $\mathcal{D}\tau$  is just  $\mathcal{D}\gamma_{0, \text{Id}', \infty}(\tau)$  (or equivalently  $\mathcal{D}\Gamma_{0, \text{Id}', 0}(\tau)$ ).<sup>35</sup> It is normalized up to a factor of  $\Gamma(0)$ ;

$$\frac{1}{\Gamma(0)} \int_{T_a} \mathcal{D}\gamma_{0, \beta', \infty}(\tau) = 1 = \frac{1}{\Gamma(0)} \int_{T_a} \mathcal{D}\Gamma_{0, \beta', 0}(\tau) \quad (\text{B.34})$$

but

$$\int_{T_a} \mathcal{D}\gamma_{\alpha, \beta', \infty}(\tau) = \frac{\Gamma(\alpha)}{\text{Det}\beta'^\alpha} = \int_{T_a} \mathcal{D}\Gamma_{\alpha, \beta', 0}(\tau) . \quad (\text{B.35})$$

**Proposition B.1** Let  $L \in \mathcal{L}(T_a, T_a)$  be a linear map and  $L^* \in \mathcal{L}(T'_a, T'_a)$  its formal adjoint. The primitive gamma integrator is invariant under  $\tau \mapsto L\tau$  (loosely,  $\mathcal{D}(L\tau) = \mathcal{D}\tau$ ).

<sup>34</sup>This definition is somewhat modified from the original definition in [6].

<sup>35</sup>Notice that  $\int_{T_a} \mathcal{D}\gamma_{0, \text{Id}', \infty}(\tau) = 1$  is the functional analog of the formal, normalized integral  $\frac{1}{\Gamma(0)} \int_0^\infty e^{-u} d(\ln u) := 1$ . In other words,  $\mathcal{D}\tau \sim du/\Gamma(0)u$ .

*Proof.*

$$\begin{aligned}
1 \stackrel{\tau' \rightarrow 0}{=} \int_{T_a} e^{i\langle \tau', L\tau \rangle - \langle \text{Id}', L\tau \rangle} \mathcal{D}(L\tau) &= \int_{T_a} e^{i\langle L^* \tau', \tau \rangle - \langle L^* \text{Id}', \tau \rangle} \mathcal{D}(L\tau) \\
&= \int_{T_a} e^{i\langle \tilde{\tau}', \tau \rangle - \langle \tilde{\text{Id}}', \tau \rangle} \mathcal{D}(L\tau) \\
&= \int_{T_a} e^{i\langle \tilde{\tau}', \tau \rangle - \langle \tilde{\text{Id}}', \tau \rangle} \mathcal{D}\tau \stackrel{\tau' \rightarrow 0}{=} 1 \quad (\text{B.36})
\end{aligned}$$

Hence,

$$\int_{T_a} F_\mu(L\tau) \mathcal{D}(L\tau) = \int_{T_a} F_\mu(L\tau) \mathcal{D}\tau. \quad (\text{B.37})$$

□

Evidently, whereas the  $\mathcal{D}x$  integrator is the infinite dimensional analog of the translation invariant measure on  $\mathbb{R}^n$ , the  $\mathcal{D}\tau$  integrator is the analog of the scale invariant measure on  $\mathbb{R}_+^n$  defined to be the identity component of  $GL(n, \mathbb{R})$ . This scale invariance is responsible for the prominent role played by the gamma integrator in solving partial differential equations. Or, reversing the interpretation, the gamma integrator (which is dictated by conjugacy) is responsible for the asymptotic ‘time’ behavior of parabolic and (bounded) elliptic/hyperbolic second order differential equations.

Put  $B_a = X_a \times T_a$ . For  $\Theta_X$  Gaussian and  $\Theta_T$  gamma, use the relation for conjugate integrators to get

$$\begin{aligned}
\int_{B_a} \Theta_{X|T}(x|\tau, \cdot) \mathcal{D}_{\Theta_{X|T}, Z_{X|T}} x|\tau &= \int_{B_a} \Theta_{X|T}(x|\tau, \cdot) \frac{\Theta_T(\tau, \cdot)}{Z_{T'}} \mathcal{D}_{\Theta_B, Z_B} b \\
&\propto \int_{B_a} \Theta_{S_s(T)|X}(S_s(\tau)|x, \cdot) \frac{\Theta_X(x, \cdot) \Theta_T(\tau, \cdot)}{Z_{T'}} \mathcal{D}_{\Theta_B, Z_B} b. \quad (\text{B.38})
\end{aligned}$$

This suggests that integrals of conditional functionals on  $X_a \times T_a$  be understood as

$$\int_{B_a} F_\mu(x|\tau) \mathcal{D}_{\Theta_{X|T}, Z_{X|T}} x|\tau = \int_{B_a} \tilde{F}_\mu(S_s(\tau), x) \mathcal{D}\omega_{\bar{x}, Q}(x) \mathcal{D}\gamma_{\alpha, \beta', \bar{\tau}_\partial}(\tau) \quad (\text{B.39})$$

when  $S_s(\tau)$  is a sufficient statistic for the integrator family characterized by  $\Theta_X$ . This is just a specialization of the solution strategy (3.14), and it plays a prominent role in the solution of differential equations.

#### B.0.4 Poisson family

Take the lower gamma integrator and regularize by replacing  $\gamma(\alpha, \bar{\tau}_\partial)$  with the regularized lower incomplete gamma function  $P(\alpha, \bar{\tau}_\partial) := \gamma(\alpha, \bar{\tau}_\partial)/\Gamma(\alpha)$ . Restrict to the

case  $\alpha = n \in \mathbb{N}$ ,  $\beta' = \text{Id}'$ , and  $\overline{\tau}_\partial = \tau_b \in \mathbb{R}_+$ . Note that, for  $N \in \text{Pois}(\tau_b)$  a Poisson random variable, we have

$$Pr(N < n) = \sum_{k < n} e^{-\tau_b} \frac{(\tau_b)^k}{k!} . \quad (\text{B.40})$$

Hence,

$$Pr(N \geq n) = \sum_{k=n}^{\infty} e^{-\tau_b} \frac{(\tau_b)^k}{k!} = P(n, \tau_b) = \frac{1}{\Gamma(n)} \int_{T_a} \mathcal{D}\gamma_{n, \text{Id}', \tau_b}(\tau) \quad (\text{B.41})$$

which, in particular, implies

$$\frac{1}{\Gamma(0)} \int_{T_a} \mathcal{D}\gamma_{0, \text{Id}', \tau_b}(\tau) = \sum_{k=0}^{\infty} e^{-\tau_b} \frac{(\tau_b)^k}{k!} . \quad (\text{B.42})$$

On the other hand,

$$e^{-\tau_b} \frac{(\tau_b)^k}{k!} = \frac{e^{-\tau_b}}{k!} \int_0^{\tau_b} \cdots \int_0^{\tau_b} d\tau_1, \dots, d\tau_k . \quad (\text{B.43})$$

Not surprisingly,  $\text{Pois}(\tau_b)$  is closely related to the restricted gamma integrator which motivates the following definition:

**Definition B.5** Let  $T_a$  be the space of  $L^{2,1}$  pointed functions  $\tau : [t_a, t_b] \rightarrow \overline{\mathbb{R}_+} := \mathbb{R}_+ \cup \{0\}$  endowed with a gamma integrator with  $\alpha = n \in \mathbb{N}$  and  $\langle \beta', \tau \rangle = \lambda \langle \text{Id}', \tau \rangle$  such that  $\lambda \in \mathbb{C}_+$ .<sup>36</sup> The Poisson integrator  $\mathcal{D}\pi_{n, \lambda, \tau_b}(\tau)$  is characterized by

$$\Theta(\tau, \tau') = \exp\{i \langle \tau', \tau \rangle\} \ ; \ Z_{n, \lambda, \tau_b}(\tau') = \frac{P(n, \lambda \tau_b)}{\text{Det}(\text{Id}' - \frac{i}{\lambda} \tau')} . \quad (\text{B.44})$$

The Poisson integrator is defined in terms of the primitive gamma integrator by

$$\mathcal{D}\pi_{n, \lambda, \tau_b}(\tau) := \lambda^n \tau^n e^{-\langle \text{Id}', \tau \rangle} \mathcal{D}\tau . \quad (\text{B.45})$$

Note the normalization

$$\int_{T_a} \mathcal{D}\pi_{0, \lambda, \tau_b}(\tau) = 1 \quad (\text{B.46})$$

and

$$\int_{T_a} \mathcal{D}\pi_{n, \lambda, \tau_b}(\tau) = P(n, \lambda \tau_b) . \quad (\text{B.47})$$

Given a functional  $F_\mu(\tau)$  integrable with respect to  $\mathcal{D}\tau$ , define its Poisson average  $\langle F \rangle_{\lambda, \tau_b}$  by

$$\langle F \rangle_{\lambda, \tau_b} := \sum_{n=0}^{\infty} (-1)^n \int_{T_a} F_\mu(\tau) \mathcal{D}\pi_{n, \lambda, \tau_b}(\tau) . \quad (\text{B.48})$$

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<sup>36</sup> $\mathbb{C}_+ := \mathbb{R}_+ \times i\mathbb{R}$  is the right-half complex plane, i.e.  $z \in \mathbb{C}$  such that  $\text{Re}(z) > 0$ .

Now let  $L_n : T_a \rightarrow \overline{\mathbb{R}_+^n}$  by  $\tau \mapsto \boldsymbol{\tau} := \{\tau_1, \dots, \tau_n\}$  where  $\tau_i = \langle \tau'_i, \tau \rangle = \langle L_i^*(\text{Id}'), \tau \rangle$  such that  $0 \leq \tau_1 \leq \dots \leq \tau_n \leq \tau_b$  and  $\sum_{i=1}^n \tau_i = \tau_b$ . Then under  $L_n$ ,  $F_\mu(\tau) \mapsto F_\mu(L_n(\tau)) =: F(\boldsymbol{\tau})$ , and using obvious notation for the measure in  $\overline{\mathbb{R}_+^n}$

$$\int_{T_a} F_\mu(L_n(\tau)) \mathcal{D}\pi_{n,\lambda,\tau_b}(\tau) \rightarrow e^{-\lambda\tau_b} \frac{(-1)^n \lambda^n}{n!} \int_{\overline{\mathbb{R}_+^n}} \theta(\boldsymbol{\tau}_b - \boldsymbol{\tau}) F(\boldsymbol{\tau}) d\boldsymbol{\tau} \quad (\text{B.49})$$

where the symmetry factor  $(-1)^n/n!$  comes from the phase  $e^{\pi i}$  (associated with the underlying complex nature of  $\tau(t)$ ) and the counting factor associated with interchanging components of  $\boldsymbol{\tau}$ .

So, if  $F_\mu(\tau)$  happens to be such that  $F_\mu(L_n(\tau))$  is given for all  $n$ , then

$$\langle F \rangle_{\lambda,\tau_b} \rightarrow e^{-\lambda\tau_b} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\overline{\mathbb{R}_+^n}} \theta(\boldsymbol{\tau}_b - \boldsymbol{\tau}) F(\boldsymbol{\tau}) d\boldsymbol{\tau} . \quad (\text{B.50})$$

In particular, if  $F_\mu(\tau) = 1$ ,

$$\langle Id \rangle_{\lambda,\tau_b} \rightarrow e^{-\lambda\tau_b} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\overline{\mathbb{R}_+^n}} \theta(\boldsymbol{\tau}_b - \boldsymbol{\tau}) d\boldsymbol{\tau} = e^{-\lambda\tau_b} \sum_{n=0}^{\infty} \frac{(\lambda\tau_b)^n}{n!} = P(0, \lambda\tau_b) = 1 . \quad (\text{B.51})$$

On the other hand,

$$\langle Id \rangle_{\lambda,\tau_b} = \sum_{n=0}^{\infty} (-1)^n \int_{T_a} \mathcal{D}\pi_{n,\lambda,\tau_b}(\tau) = \sum_{n=0}^{\infty} (-1)^n P(n, \lambda\tau_b) . \quad (\text{B.52})$$

There is no inconsistency here because

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n P(n, \lambda\tau_b) &= [P(0, \lambda\tau_b) - P(1, \lambda\tau_b)] + [P(2, \lambda\tau_b) - P(3, \lambda\tau_b)] + \dots \\ &= e^{-\lambda\tau_b} \sum_{2n=0}^{\infty} \frac{(\lambda\tau_b)^{2n}}{2n!} \\ &= P(0, \lambda\tau_b) . \end{aligned} \quad (\text{B.53})$$

This definition of a Poisson functional integral agrees with the definition in [1] and, hence, gives an alternative characterization of a Poisson integrator. It is already known that such functional integrals give solutions to first-order operator differential equations. The fact that the Poisson integrator is composed within the family of gamma integrators indicates that gamma integrators will play a role in solving first-order operator differential equations with constraints.

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